PARTIAL DIFFERENTIAL EQUATIONS

- A Quick Guide -

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Preface

Greetings,

Partial Differential Equations: A Quick Guide is based on my lecture notes from MA411: Topics in Differential Equations - Partial Differential Equations with professor Evan Randles at Colby. The contents are somewhat based on Farlow's *Partial Differential Equations for Scientists and Engineers*.

Enjoy!

Contents

Chapter 1

Overview and Classification

1.1 What in the world is a PDE?

We shall begin with what PDEs are.

Definition 1.1.1. A partial differential equation (PDE) is an equation relating a function of several variables $\psi(t, \vec{x})$ to its partial derivatives: $\partial_{x_1}\psi$, $\partial^2_{x_1 x_2}\psi$, etc.

A note on notation:

$$\frac{\partial^2 \psi}{\partial x_1\, \partial x_2} \equiv \partial^2_{x_1 x_2}\psi \equiv \partial_{x_1}\partial_{x_2}\psi.$$

1.2 Some notable examples

Let us look at a couple of famous PDEs:

Example 1.2.1. Laplace Equation:

$$\Delta\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0.$$

Example 1.2.2. Poisson's Equation:

$$\Delta\psi = \nabla^2\psi = F(x, y, z)$$

We take note of the **Laplacian** or the **Laplacian operator**:

$$\boxed{\Delta\psi \equiv \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}}$$

The Laplacian operator takes a function ψ linearly to another function $\nabla^2\psi$. The Laplacian is one of the most important objects in mathematics, as it touches probability theory, potential theory, partial differential equations, mathematical physics, harmonic analysis, number theory, etc.

Another note on notation: the symbols Δ and ∇^2 will be used interchangeably in this text. The ∇^2 represents the divergence of the gradient.

Let us look at some more examples to see the ubiquity of the Laplacian in PDEs:

Example 1.2.3. The heat equation:

$$\frac{\partial \psi}{\partial t} = \nabla^2 \psi.$$

The heat equation describes heat transfer over time. But there is also a connection between the heat equation and probability theory. In particular, the Gaussian function:

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

solves the heat equation.

Example 1.2.4. The wave equation:

$$\frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi.$$

The wave equation describes physical vibrations. The second t-derivative in the equation is strongly correlated to Newton's second law of motion.

Example 1.2.5. The Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(t, \vec{x})\psi.$$

One can hardly talk about PDEs without mentioning the Schrödinger equation. There is a strong resemblance between the Schrödinger equation and the wave equation. Of course, this is no coincidence, as the Schrödinger equation is postulated based on a description of a harmonic oscillator.

Our next example does not include the Laplacian operator.

Example 1.2.6. The telegraphic equation:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \alpha \frac{\partial \psi}{\partial t} + \beta \psi.$$

The telegraphic equation describes the transfer of information.

1.3 Vocabulary

- The function ψ appearing in a given PDE is called the "dependent variable."
- The variables $t, x_1, x_2, \ldots$ are called "independent variables."

1.4 Our goals

Our goal is, given a PDE, to find a sufficiently differentiable function which satisfies it that is subject to **boundary** and **initial** conditions.

1.5 Our plan

Here are the key concepts we will explore in this text:

- Modeling: Formulate same physical problem in terms of PDEs.
- Learn how to solve (some) PDEs, subjection to initial conditions and boundary conditions. This means we will be looking at ideas like:
  - Separation of variables, in order to reduce a PDE into a system of ODEs.
  - Integral transforms, in order to reduce the number of independent variables.
  - Change of coordinates, in order to change a complicated PDE into another one which is easier to solve.
  - Eigenfunction expansion, which generally goes under the Sturm-Liouville theory.
  - Numerical methods, as most PDEs cannot be solved analytically.

1.6 Classification

- The order of a PDE is the highest order of partial derivatives appearing (non-trivially) in the PDE.

  Example 1.6.1.

  $$\frac{\partial \psi}{\partial t} = \nabla^2 \psi$$

  is a second-order PDE.

  Example 1.6.2.

  $$\frac{\partial \psi}{\partial t} = \partial_x^4 \psi$$

  - the biharmonic heat equation, is a fourth-order PDE.

- Linearity: A PDE is linear if the function ψ and its derivatives appear in a linear way.

  Example 1.6.3. All second-order linear PDEs in 2 variables are of the form:

$$\boxed{A\frac{\partial^2\psi}{\partial x^2} + B\frac{\partial^2\psi}{\partial x\,\partial y} + C\frac{\partial^2\psi}{\partial y^2} + D\frac{\partial\psi}{\partial x} + E\frac{\partial\psi}{\partial y} + F\psi = G}$$

Note: define

$$L[\psi](x,y) = A\frac{\partial^2\psi}{\partial x^2} + B\frac{\partial^2\psi}{\partial x\,\partial y} + C\frac{\partial^2\psi}{\partial y^2} + D\frac{\partial\psi}{\partial x} + E\frac{\partial\psi}{\partial y} + F\psi$$

then we get

$$L[u] = G.$$

We get a linear map $L : \psi \to L[\psi]$. So, for $\gamma, \sigma \in \mathbb{R}$

$$L[\gamma u + \sigma v)] = \gamma L[u] + \sigma L[v].$$

This observation justifies the moniker "linear." Next, we say that $L[\psi] = G$ is **homogeneous** if $G = 0$. The equation is **inhomogeneous** if $G(x,y) \neq 0$ for some x, y.

If A, B, C, D, E, F are constants, then $L[\psi] = G$ is said to be a **constant-coefficient** equation. Otherwise (at least one of A, B, C, D, E, F is a function of x, y in some non-trivial way), it is said to have **variable coefficients**.

Example 1.6.4. Classify: $u_t = \sin t u_{xx}$.

It is a linear PDE, $A = \sin t$, $B = C = D = F = 0$, $E = -1$, $G = 0$, variable coefficient, and homogeneous.

Example 1.6.5. Classify: $u_{xx} - \sin u = 0$.

Not linear.

Example 1.6.6. Classify: $xu_x - yu_y = 0$.

First-order homogeneous linear PDE with variable coefficients.

Note: Linear PDEs are quite well understood. Notable mathematicians who established theories of linear PDEs: Ehenpres(?), Hille, Browder, Soboher, Nash, Nierenburd, Friedmann, Schwartz, Hormander (Fields, 1962), Gardiy.

Note: Constant coefficient equations are **much easier** to solve than variable coefficient equations, because Fourier analysis makes a lot of the constant coefficient problems easy.

Note: Non-linear equations are really hard, and there is no general theory. Each type of non-linear problem demands its own special techniques (well, if they exist at all).

1.7 Types of second order linear PDE

Parabolic: $L[\psi] = G$ is said to be parabolic if $B^2 - 4AC = 0$ (A, B, C don't have to be constant coefficients - so the PDE can be parabolic in some region and not elsewhere).

Example 1.7.1. The heat equation

$$u_t = u_{xx}$$

is a parabolic equation, because $A = 1, E = -1, B = C = 0$.

Elliptic: $L[\psi] = G$ is elliptic if $B^2 - 4AC < 0$.

Example 1.7.2. Laplace's equation

$$\delta u = u_{xx} + u_{yy} = 0$$

is elliptic, because $A = C = 1, B = 0$.

Hyperbolic: if $B^2 - 4AC > 0$.

Example 1.7.3. The wave equation:

$$u_{tt} = u_{xx}$$

is hyperbolic, because $B = 0, A = 1, C = -1$.

1.8 Lesson 1: Selected Problems & Solutions

Problem (3). If $u_1(x,t)$ and $u_2(x,t)$ satisfy $L[u] = G$, then is it true that the sum satisfies it? If yes, show.

Solution (3). Yes. We have established, in class, that if we define $L : u \to L[u]$ where

$$L[u] = Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = G,$$

then $L[u]$ is a linear map, which can be readily shown:

$$\begin{aligned} L[\mu u_1 + \nu u_2] &= \mu(Au_{1xx} + Bu_{1xt} + Cu_{1tt} + Du_{1x} + Eu_{1t} + Fu_1) \\ &\quad + \nu(Au_{2xx} + Bu_{2xt} + Cu_{2tt} + Du_{2x} + Eu_{2t} + Fu_2) \\ &= \mu L[u_1] + \nu L[u_2]. \end{aligned}$$

So, the sum of u_1 and u_2 also satisfies $L[u] = G$.

Problem (4). Probably the easier of al PDEs to solve is the equation

$$\frac{\partial u(x,y)}{\partial x} = 0.$$

Can you solve this equation? Find all functions $u(x,y)$ that satisfy it.

Solution (4). The PDE suggests that u does not depend on x. This means that $u(x,y)$ is just some function of y, i.e. $u(x,y) = \tilde{u}(y)$.

Problem (5). What about the PDE

$$\frac{\partial^2 u(x,y)}{\partial x\, \partial y} = 0?$$

Can you find all solutions $u(x,y)$ to this equation? How many are there? How does this compare with an ODE like

$$\frac{d^2 y}{dx^2} = 0$$

insofar as the number of solutions is concerned?

Solution (5). This PDE is a first-order, linear, homogeneous PDE with $B = 1, A = C = D = E = F = 0$. Since $B^2 - 4AC = 1 > 0$, the PDE is **hyperbolic**. The PDE suggests that u_y has no x-dependence. From the previous problem, we know that $u_y = f'(y)$. Taking the antiderivative with respect to y, we get

$$\boxed{u(x,y) = \int f'(y)\, dy = f(y) + g(x)}.$$

Since the variables x, y are exchangeable (by the equality of mixed partials), following the same argument starting with u_x gives the same form for $u(x, y)$.

The ODE $D^2[y] = y''(x) = 0$ is a second-order, linear, homogeneous ODE. We know that the solution space has dimension of 2:

$$\ker(D^2) = \{1, x\}.$$

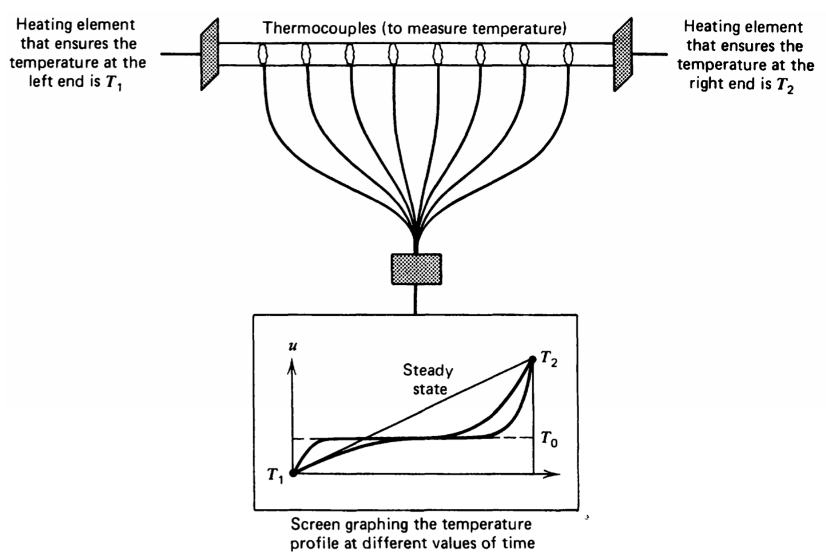
So while there are infinitely many solutions, only **two** linearly independent solutions are sufficient to generate all solutions. Whereas there are infinitely many linearly independent solutions to the PDE. We can simply generate a new (linearly independent from $f(y) + g(x)$) solution by multiplying $f(y)$ by y or $g(x)$ by x.

Chapter 2

Diffusion-type problems (parabolic equations): A study of the heat equation

2.1 An experiment

We consider a copper rod of length L, which allows heat to transfer along the rod, but is insulated in such a way that heat does not transfer laterally across/out of the rod.



At time $t = 0$, the temperature in the rod is known.

$$u(0, x) = T_0$$

The ends of the rod are placed in thermal baths which hold their temperatures fixed. So, at $x = 0$, $u(t, 0) = T_1$ and at $x = L$, $u(t, L) = T_2$ for all $t > 0$.

2.2 The Mathematical Model

This behavior is modeled by the heat equation.

$$u_t = \alpha^2 u_{xx},$$

where $\alpha \in \mathbb{R}$, determined by the thermo-character of the rod. u_t is the rate of change of temperature in time, and u_{xx} is the concavity profile in space.

Some justification for the heat equation: we look at the spatial difference quotient. For small change in x, Δx:

$$\begin{aligned}
u_{xx} &\approx \frac{u_x(t, x + \Delta x) - u_x(t, x)}{\Delta x} \\
&\approx \frac{(u(t, x + \Delta x) - u(t, x))/\Delta x - (u(t, x) - u(t, x - \Delta x))/\Delta x}{\Delta x} \\
&\approx \frac{1}{\Delta x^2}(u(t, x + \Delta x) + u(t, x - \Delta x) - 2u(t, x)) \\
&\approx \frac{2}{\Delta x^2}\left(\frac{u(t, x + \Delta x) + u(t, x - \Delta x)}{2} - u(t, x)\right)
\end{aligned}$$

So, $u_{xx} \propto$ the difference between the average temperatures among neighboring points and the temperature at x.

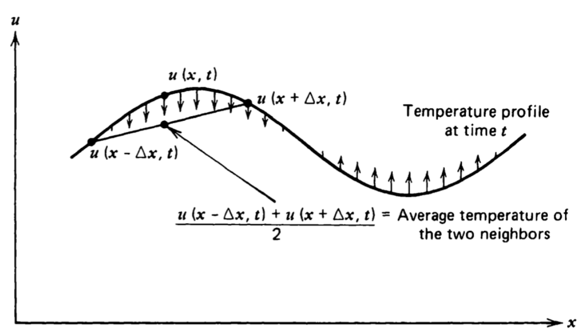


FIGURE 2.2 Arrows indicating change in temperature according to $u_t = \alpha^2 u_{xx}$.

Assume that $u_t = \alpha^2 u_{xx}$, then if $u_{xx} < 0$, then $u_t < 0$, i.e. temperature decreases in time. If $u_{xx} > 0$, then $u_t > 0$, i.e. temperature increases in time. If $u_{xx} = 0$, the temperature stays fixed.

2.3 Boundary Conditions

In contrast to ODEs, PDE have different types of constraints which are combined with the PDE to form well-posed problems, where "well-posed" means that a unique solution exists. Our conditions are often (and will almost always) be physically motivated.

Let us revisit the heat equation.

$$u_t = \alpha^2 u_{xx}, t > 0, 0 \leq x \leq L.$$

The temperatures at the ends $x = 0$ and $x = L$ are fixed T_1 and T_2 by the thermal baths, so the boundary conditions are

$$BCs = \begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} \forall t > 0.$$

Here "boundary" refers to the boundary of $[0, L]$.

2.4 Initial Conditions

Our problem also involves evolution in time, we have an initial condition of the form

$$u(x,0) = T_0 \text{ or } u(x,0) = u_0(x) \forall x \in [0, L]$$

where T_0 is the initial constant temperature of the rod and $u_0(x)$ is the initial temperature which is allowed to vary (some spatial distribution). All together, we form an initial boundary value problem, an IBVP of the form

$$\begin{cases} u_t = \alpha^2 u_{xx}, t > 0, x \in [0, L] \\ u(0,t) = T_1, \forall t > 0 \\ u(L,t) = T_2, \forall t > 0 \\ u(x,0) = T_0, \forall x \in [0, L] \end{cases}$$

2.5 A Couple of Variants

2.5.1 Lateral Heat Loss

This allows for heat to be transferred laterally into the rod according to Newton's law of cooling. So the new heat equation is

$$u_t = \alpha^2 u_{xx} - \beta(u - u_0), \beta > 0$$

where u_0 is the outside temperature.

2.5.2 Internal Heat Source

If, by some non-diffusive heat source, heat is added into the rod at (t, x), the equation is

$$u_t = \alpha^2 u_{xx} + f(x,t)$$

where $f(x,t)$ is the heat added to the rod, internally. This PDE is *inhomogeneous.*

2.5.3 Diffusion-convection Equation

$$u_t = \alpha^2 u_{xx} - v u_x$$

If u describes the amount (not heat) pollutant, then the term $-vu_x$ describes the flow of additional pollutant introduced by the moving particles.

2.5.4 Variable-coefficients case

When the thermal make up of the rod (its thermal character) is allowed to vary according to a variable diffusivity coefficient, i.e. $\alpha \to \alpha(x)$, then the relevant heat equation is

$$u_t = \alpha^2(x) u_{xx}.$$

So, let's say

$$\alpha(x) = \begin{cases} \alpha_{Copper}, x \in [0, L/2] \\ \alpha_{Bronze}, x \in [L/2, L] \end{cases}$$

Chapter 3

Other types of Boundary Conditions

3.1 Type 1

Let's revisit the original heat equation: $u_t = \alpha^2 u_{xx}$. If we force the rod ends to have time-dependent temperatures: $g_1(t)$ and $g_2(t)$ at $x = 0$ and $x = L$ respectively, then our boundary conditions are

$$BCs = \begin{cases} u(0,t) = g_1(t) \\ u(L,t) = g_2(t) \end{cases} \forall t > 0.$$

If instead we're studying the heat flow on a circular plate, i.e., where $u = u(t,t,\theta)$, and the heat EQ is

$$u_t = \alpha^2 \nabla^2 u = \alpha^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right).$$

Here, the boundary conditions look like $u(t, r_0, \theta) = g(t, \theta)$, i.e. we force the disk to have temperature $g(t,\theta)$ along the boundary.

3.2 Type 2 (more realistic)

We take into account heat transfer to rod ends via thermal bath. Suppose that our rod is placed in bath (liquid) at each end of temperature $g_1(t)$ and $g_2(t)$ respectively.

In view of Newton's law of cooling, the heat flux at a rod end is $h(u(t,0) - g_(1))$ at $x = 0$ and $h(u(t,L) - g_2(t))$ at $x = L$, and h is some constant. Next, we introduce Fourier's law of heat flux (empirical):

$$\frac{\partial u}{\partial n} = k \times \text{Heat flux}$$

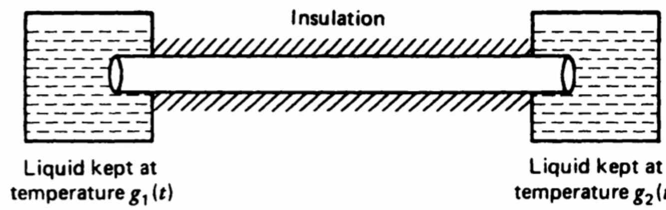


FIGURE 3.3 Convection cooling at the boundaries.

where n is the *inward normal* direction to the boundary, and $k \in \mathbb{R}$. At $x = 0$:

$$\begin{cases} \frac{\partial u}{\partial n} = u_x(t,0) = kh(u(t,0) - g_1(t)), x = 0 \\ \frac{\partial u}{\partial n} = -u_x(t,L) = kh(u(t,L) - g_2(t)), x = L \end{cases}.$$

So, the BC for Type 2 is the following

$$\begin{cases} u_x(t,0) = kh(u(t,0) - g_1(t)) \\ u_2(t,L) = -kh(u(t,L) - g_2(t)) \end{cases}$$

The 2-D plate analogue is the following. We require (since r is outward)

$$-\frac{\partial u}{\partial r} = -kh(u(t, r_0, \theta) - g(t,\theta))$$

where $g(t,\theta)$ is the temperature of the bath surrounding the plate.

3.3 Type 3: Flux specified - including isolated boundaries

The rod ends are insulated, i.e., no heat flows in or out of the rod ends. So the boundary conditions are

$$u_x(0,t) = u_x(L,t) = 0 \forall 0 < t < \infty.$$

In two variables (a disk), the analogous BC is

$$u_r(t, r_0, \theta) = 0 \forall 0 < t < \infty, 0 \leq \theta \leq 2\pi.$$

3.4 Type 4: Mixed

We can mix BCs. Suppose that one end of the rod has zero flux condition (type 3) and the other end is submerged in a liquid (type 2).

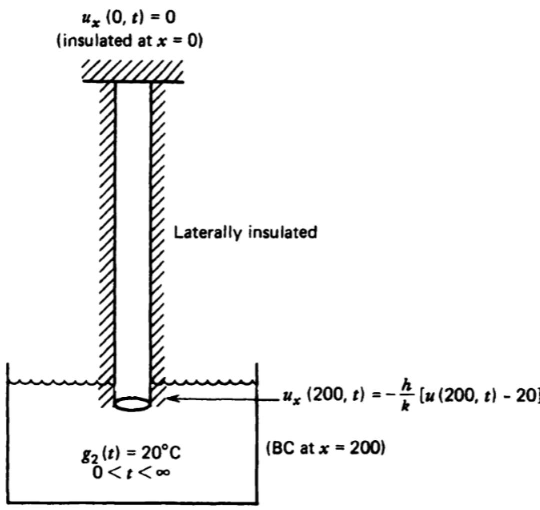


FIGURE 3.6 Initial-boundary-value problem.

So, the IBVP is

$$\begin{cases} u_t = \alpha^2 u_x x \\ u_x(t, L) = 0 \\ u_x(t, 0) = -\lambda(u(t) - g_1(t)) \forall t > 0 \\ u(0, x) = u_0(x) \forall 0 \leq x \leq L \end{cases}$$

Chapter 4

Derivation of the Heat Equation

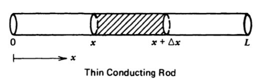


FIGURE 4.3 Thin conducting rod.

Main idea: Conservation of (Heat) Energy. Assumptions:

1. The rod is a thermally homogeneous material

2. The temperature is constant across all cross-sections

3. The rod is laterally insulated (no heat loss laterally)

Using conservation of energy, we have the following: the change in thermal energy in the cross section x to $x + \Delta x$ should be equal to the flux of the heat through the "ends" at x and $x + \Delta x$ plus any external heat produced by some source (e.g. heat element). Some physical constants:

1. C: thermal capacity of the rod

2. ρ: density of the material of the rod

3. A: area of cross section

4. k: thermal conductivity

Total heat inside is

$$\int_{x}^{\Delta x + x} c\rho A u(s,t)\, ds.$$

The flux through the ends is

$$kA(u_x(x+\Delta x,t)-u_x(x,t)).$$

The external energy is

$$A\int_x^{x+\Delta x} f(t,s)\,ds$$

where $f(t,s)$ is the energy added at time t and $x \le s \le x+\Delta x$. All together

$$\frac{d}{dt}\int_x^{\Delta x+x} c\rho A u(s,t)\,ds = kA(u_x(x+\Delta x,t)-u_x(x,t)) + A\int_x^{x+\Delta x} f(t,s)\,ds.$$

Assuming that u is nice enough, that

$$\frac{d}{dt}\int_x^{\Delta x+x} c\rho A u(s,t)\,ds = \int_x^{\Delta x+x} c\rho A u_t(s,t)\,ds.$$

Also, the MVT for integrals says that if G is continuous on the interval $[a,b]$ then $\exists c \in [a,b]$ such that

$$\int_a^b G(s)\,ds = G(c)(b-a).$$

Therefore $\exists \chi \in [x,\Delta x]$ such that

$$\int_x^{\Delta x+x} c\rho A u_t(s,t)\,ds = c\rho A u_t(t,\chi)\Delta x$$

and $\exists \eta \in [x,\Delta x]$ such that

$$A\int_x^{x+\Delta x} f(t,s)\,ds = Af(t,\eta)\Delta x.$$

Combining all of these gives $\forall t>0, \exists \chi,\eta \in [x,\Delta x]$ such that

$$\begin{aligned} c\rho A u_t(t,\chi)\Delta x &= kA(u_x(t,x+\Delta x)-u_x(t,x)) + Af(t,\eta)\Delta x \\ u_t(t,\chi) &= \frac{k}{\rho c}\frac{u_x(t,x+\Delta x)-u_x(t,x)}{\Delta x} + \frac{1}{c\rho}f(t,\eta). \end{aligned}$$

As $\Delta x \to 0$, $\eta,\chi \to x$

$$\begin{aligned} u_t(t,x) &= \frac{k}{\rho c}u_{xx}(x,t) + \frac{1}{c\rho}f(t,x) \\ u_t(t,x) &= \frac{k}{\rho c}u_{xx}(x,t) + F(t,x) \\ u_t(t,x) &= \alpha^2 u_{xx}(x,t) + F(t,x), \end{aligned}$$

where

$$\begin{aligned} \alpha^2 &= \frac{k}{\rho c} \\ F(t,x) &= \frac{1}{c\rho}f(t,x). \end{aligned}$$

Chapter 5

Separation of Variables - First method of solution

Main idea: If the IBVP is posed on a rectangle, e.g. $t > 0, x \in [0, L]$, and the PDE is linear, it is often the case that this method reduces the PDE into ODEs.

5.1 Example: The heat equation

$$u_t = \alpha^2 u_{xx}, t > 0, x \in [0, 1]$$

We shall accompany this with so-called linear homogeneous BCs:

$$\begin{aligned} \alpha u(t, 0) + \beta u_x(t, 0) = 0 \\ \gamma u(t, 1) + \delta u_x(t, 1) = 0. \end{aligned}$$

In fact, we specify further to assume

$$u(t, 0) = 0 = u(t, 1) \forall t > 0.$$

We make an ansatz that solutions are of the form

$$u(t, x) = T(t)X(x).$$

(Maybe not solutions but builidng blocks of solutions). Plug into the PDE, we get

$$u_t(t, x) = T'(t)X(x) = \alpha^2 \partial_x^2(u(t, x)) = \alpha^2 T(t)X''(x).$$

Separating variables gives

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} \forall t > 0, x \in [0, 1].$$

For this equation to hold for all independent t and x, we must have that

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \text{Const}\forall t > 0, x \in [0,1].$$

This immediately gives two ODEs connected by a constant k:

$$\begin{cases} T(t) = \alpha^2 k T(t) \\ X''(x) = kX(x) \end{cases}.$$

By solving these equations, we hope to learn something aboaut k. The solution to the first solution is obvious:

$$K(t) = Ae^{\alpha^2 kt}\forall t > 0.$$

For physically reasonable solutions, we expect that the limit as $t \to \infty$ of $u(t,x) = 0$ and so, $T(t) \not\to \infty$ as $t \to \infty$, this forces $k < 0$. Thus, we write $k = -\lambda^2$, $\lambda \in \mathbb{R}$, and denote

$$u_\lambda(t,x) = T(t)X(x) = X(x)Ae^{-\alpha^2\lambda^2 t}.$$

Next, the spatial ODE gives

$$X''(x) + \lambda^2 X(x) = 0.$$

A general solution for this equation is

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x).$$

By absorbing multiplcative constants

$$u_\lambda(t,x) = e^{-\alpha^2\lambda^2 t}\left(A\sin(\lambda x) + B\cos(\lambda x)\right).$$

Though we still don't know what λ is, let us force this solution to satisfy the boundary conditions to learn more. Since the boundary conditions require that $u(t,0) = 0 = u(t,1)$, we require that

$$\begin{cases} B = 0 \\ \lambda = \pm n\pi \end{cases}$$

where $n \in \mathbb{N}$, for non-trivial solutions (where $A \neq 0$). Thus, with separation of variables, we find that

$$u_n(t,x) = A_n e^{-(n\alpha\pi)^2}\sin(n\pi x).$$

This is a solution for any $n \in \mathbb{N}$ and $A_n \in \mathbb{R}$. Just to be sure that we haven't made an error, we can readily verify this solution. This is left as an exercise to the reader.

Note that we still have some "degrees of freedom" - A_n and n. So, we have established the existence of *many* solutions, for each $n \in \mathbb{N}$ and $A_n \in \mathbb{R}$. Now, we make use of the principle of superposition to generate more solutions. The principle of superposition (works for linear DEs) says that all convergent sums of solutions are solutions. More generally, for any collection $\{A_n\} \subseteq \mathbb{R}$,

$$u(t,x) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2} \sin(n\pi x)$$

is also a solution. But divergence could be a problem here. It might be that $u(t,x)$ fails to exist, or differentiation might not work. But worry not, since the presence of the term $e^{-(n\pi\alpha)^2}$ makes this series always converge. And so, we have that for any sequence $\{A_n\}$, $u(t,x)$ defined in this way solves the DE $u_t = \alpha^2 u_{xx}$ and satisfies the boundary conditions $u(t,0) = 0 = u(t,1)$. The problem of satisfying the initial condition $u(0,x) = u_0$ becomes one of finding the "right" constants A_n so that

$$u(0,x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = u_0(x).$$

The term on the left hand side is called the trigonometric series. The question now becomes whether it is possible to find the sequence $\{A_n\} \subseteq \mathbb{R}$ so that

$$u(0,x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = u_0(x).$$

Another question would be which function $u_0(x)$ can be expanded as a trigonometric series as above.

Example 5.1.1. Consider

$$u_0(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{50}\sin(201\pi x).$$

We see that

$$A_n = 0, A_2 = \frac{1}{2}, A_{201} = \frac{1}{50} \forall n \neq 2, 201.$$

Example 5.1.2. What about

$$u_0(x) = \begin{cases} x, 0 \leq x < \frac{1}{2} \\ 1-x, \frac{1}{2} < x \leq 1 \end{cases}$$

or

$$u_0(x) = 1?$$

It's clear that we must have that $u_0(0) = u_1(1) = 0$, otherwise this cannot be done. To treat otherwise, one needs a cosine term. But what if $u_0(0) = u_0(1) = 0$, but $u_0(x)$ is very bad? Suppose that this can be done. Using the property that

$$\int_0^1 \sin(n\pi x)\sin(m\pi x) = \frac{1}{2}\delta_n^m,$$

we use **Fourier's trick**: multiply both sides of the $u_0(x)$ expansion by $\sin(m\pi x)$ and integrate:

$$\begin{aligned}\int_0^1 u_0(x)\sin(m\pi x) &= \sum_{n=1}^{\infty}\int_0^1 A_n \sin(n\pi x)\sin(m\pi x)\,dx \\ &= \sum_{n=1}^{\infty} A_m \frac{1}{2}\delta_n^m.\end{aligned}$$

So this gives

$$A_m = 2\int_0^1 u_0(x)\sin(m\pi x)\,dx \forall m \in \mathbb{N}.$$

This gives a prescription for finding the sequence $\{A_m\}$ so that the expansion works. So, we might ask, given a function $u_0(x)$ with value 0 at $x = 0, 1$ and define

$$A_m = 2\int_0^1 u_0(x)\sin(m\pi x)\,dx \;\; \forall m \in \mathbb{N},$$

when does

$$u_0(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)?$$

Usually, this works perfectly, but around 1802, the mathematician DuBois Reymond found an example for which the **Fourier series** does not hold. The exact class of such functions was determined explicitly in 1962 by UCLA professor L. Carelson. The answer is $L^2[0,1]$ - square integrable functions.

Example 5.1.3. Now, let's find A_n so that

$$u_0(x) = \begin{cases} x, 0 \le x < \frac{1}{2} \\ 1 - x, \frac{1}{2} < x \le 1 \end{cases}.$$

Well,

$$\begin{aligned}A_n = 2\int_0^1 u_0(x)\sin(n\pi x)\,dx &= 2\int_0^{\frac{1}{2}} x\sin(n\pi x)\,dx + 2\int_{\frac{1}{2}}^1 (1-x)\sin(n\pi x)\,dx \\ = 2\int_0^{\frac{1}{2}} x\sin(n\pi x)\,dx &- 2\int_{\frac{1}{2}}^1 +x\sin(n\pi x)\,dx + 2\int_{\frac{1}{2}}^1 \sin(n\pi x)\,dx\end{aligned}$$

Integration by parts...

$$\int x\sin(kx)\,dx = \frac{-1}{k}x\cos(kx) - \int \frac{-\cos(kx)}{k}\,dx = \frac{1}{k^2}\sin(kx) - \frac{x\cos(kx)}{k}.$$

So

$$2\left(\frac{1}{(n\pi)^2}\sin(n\pi x) - \frac{x}{\pi n}\cos(n\pi x)\right)\Bigg|_0^{1/2} = \frac{2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right).$$

$$2\int_{\frac{1}{2}}^{1}(1-x)\sin(n\pi x)\,dx = \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right).$$

$$2\int_{\frac{1}{2}}^{1} x\sin(n\pi x)\,dx = \frac{2}{n\pi}\cos(n\pi) - \frac{2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right).$$

So, all together,

$$A_n = \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right).$$

So our series is

$$u_0(x) = \sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\sin(n\pi x)$$

This converges nicely.

Recap: to solve our IVBP, we defines

$$A_n = 2\int_0^1 u_0(x)\sin(n\pi x)\,dx, \quad n\in\mathbb{N}$$

and then (provided that things converge nicely)

$$u(t,x) = \sum_{n=1}^{\infty} A_n e^{-(\alpha\pi n)^2 t}\sin(n\pi x)$$

is **the** solution. More generally, on the interval $[0,L]$ for the same IVBP with $u(t,0) = u(t,L) = 0$ and $u(0,x) = u_0(x), x\in[0,L]$, then the solution is given by

$$u(t,x) = \sum_{n=1}^{\infty} A_n e^{-(\alpha\pi n/L)^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

where

$$A_n = \frac{2}{L}\int_0^1 u_0(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad n\in\mathbb{N}.$$

One more note: as $t\to\infty$, the solution is dominated lower order terms

$$u(t,x) \approx A_1 e^{-(\alpha\pi/L)^2 t}\sin\left(\frac{\pi x}{L}\right).$$

Chapter 6

Transforming Nonhomogeneous Boundary Conditions into Homogeneous ones

6.1 Inhomogeneous BCs to Homogeneous Ones

Question: how can we solve inhomogeneous BCS? Given $k_1, k_2 \in \mathbb{R}$ and $u_0(x)$, can we solve

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(t,0) = k_1 \\ u(t,L) = k_2 \\ u(0,x) = u_0(x) \end{cases}$$

Idea: in the steady-state, we expecet that $u_t = 0$, so $u(x,t) \to u(x) = Cx + D$. So, the steady state is

$$S(t,x) = k_1 \left(1 - \frac{x}{L}\right) + k_2 \left(\frac{x}{L}\right).$$

To solve the problem, we assume

$$u(t,x) = S(t,x) + U(t,x)$$

where $S(t,x)$ is the steady-state solution, and $U(t,x)$ is the transient solution. Let's plug $u = S + U$ into our problem to learn something about U. First,

$$\begin{aligned} u_t = S_t + U_t &= \alpha^2 (S_{xx} + U_{xx}) \\ U_t &= \alpha^2 U_{xx}. \end{aligned}$$

Using the BCs in out IBVP,

$$k_1 = u(t,0) = S(t,0) + U(t,0) = k_1 + U(t,0).$$

So

$$U(t,0) = 0.$$

And similarly,

$$U(t,L) = 0.$$

Finally,

$$u_0(x) = u(0,x) = S(0,x) + U(0,x)$$

and hence

$$U(0,x) = u_0(x) - S(0,x).$$

So, summary: $u(t,x)$ satisfies the IBVP $\iff$

$$u(t,x) = S(t,x) + U(t,x) = k_1\left(1 - \frac{x}{L}\right) + k_2\left(\frac{x}{L}\right) + U(t,x)$$

where $U(t,x)$ solves the auxiliary IBVP:

$$\begin{cases} U_t = \alpha^2 U_{xx} \quad t > 0, 0 < x < L \\ U(t,0) = 0 \;\; t > 0 \\ U(t,L) = 0 \\ U(0,x) = u_0(x) - S(0,x) \quad x \in [0,L] \end{cases},$$

which we know how to solve using separation of variables and Fourier series.

6.2 Time Varying BCs into Zero BCs

$$\begin{cases} u_t = \alpha^2 u_{xx} \quad , t > 0, 0 < x < L \\ \alpha_1 u(t,0) + \beta_1 u_x(t,0) = g_1(t) \\ \alpha_2 u(t,L) + \beta_2 u_x(t,L) = g_2(t) \\ u(0,x) = u_0(x) \quad , x \in [0,L] \end{cases},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, g_1(t), g_2(t), u_0(x)$ are all known. To solve this, we push forwardour idea of steady-state-ish and transient solutions. We assume that

$$u(t,x) = S(t,x) + U(t,x)$$

where

$$S(t,x) = A(t)\left(1-\frac{x}{L}\right) + B(t)\frac{x}{L}$$

Can we find $A(t)$ and $B(t)$ in terms of $\alpha_1, \alpha_2, \beta_1, \beta_2, g_1(t), g_2(t), u_0(x)$? We want to choose $S(t,x)$ so that it absorbs all of the *complicated* nature of the BCs for $u(t,x)$. So, we can make S satisfy $u's$ BCs.

$$\begin{aligned} S(t,0) &= A(t) \\ S_x(t,0) &= \frac{B(t)-A(t)}{L} = S_x(t,L) \\ S(t,L) &= B(t). \end{aligned}$$

So we have

$$\begin{aligned} \alpha_1 S(t,0) + \beta_1 S_x(t,0) &= g_1(t) \\ \alpha_1 A(t) + \frac{\beta_1(B(t)-A(t))}{L} &= g_1(t) \end{aligned}$$

and

$$\begin{aligned} \alpha_2 S(t,L) = \beta_2 S_x(t,L) &= g_2(t) \\ \alpha_2 B(t) + \frac{\beta_2(B(t)-A(t))}{L} &= g_2(t). \end{aligned}$$

Rewriting gives

$$\Gamma\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} \alpha_1 - \frac{\beta_1}{L} & \frac{\beta_1}{L} \\ -\frac{\beta_2}{L} & \alpha_2 + \frac{\beta_2}{L} \end{pmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

So

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \Gamma^{-1}\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{pmatrix}\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

Of course, this requires Γ to be invertible, i.e,

$$\det(\Gamma) = \left(\alpha_1 - \frac{\beta_1}{L}\right)\left(\alpha_2 + \frac{\beta_2}{L}\right) + \frac{\beta_1}{L}\frac{\beta_2}{L} = \alpha_1\alpha_2 + \frac{\alpha_1\beta_2}{L} - \frac{\alpha_2\beta_1}{L} \neq 0.$$

Assuming this is true

$$S(t,x) = (\rho_{11}g_1(t) + \rho_{12}g_2(t))\left(1-\frac{x}{L}\right) + (\rho_{21}g_1(t) + \rho_{22}g_2(t))\frac{x}{L}$$

Now, let us see what this implies for U.

$$u_t = S_t + U_t = U_t = \alpha^2(S_{xx} + U_{xx}) = \alpha^2 U_{xx}.$$

So, once again

$$U_t = \alpha^2 U_{xx} - S_t.$$

Also, by an easy calculation, we have that u satisfying the inhomogeneous BCs implies

$$U(t,0) = U(t,L) = 0.$$

And further, that

$$U(0,x) = u_0 - S(0,x)$$

is just a linear function of x. So, by setting

$$S(t,x) = (\rho_{11}g_1(t) + \rho_{12}g_2(t))\left(1 - \frac{x}{L}\right) + (\rho_{21}g_1(t) + \rho_{22}g_2(t))\frac{x}{L},$$

we see that

$$u(t,x) = U(t,x) + S(t,x)$$

satisfies a IBVP where the heat equation is homogeneous, but the BCs are very complicated compared to the new inhomogeneous heat IBVP.

$$\begin{cases} U_t = \alpha^2 U_{xx} - S_t \\ U_x(L,t) = 0 \\ U(t,L) = 0 \\ U(0,x) = u_0(x) - S(0,x) \end{cases}.$$

Moral: problems with complicated BCs can often be transformed into equivalent problems with simple BCs at the cost of making the PDE more complicated.

Question: Under what conditions on the first IBVP is our new IBVP homogeneous? The answer is that a sufficient condition is for g_1, g_2 to be constant. In this realm, $S(t,x) = S(x)$ nad is thus a “real” steady-state solution.

Chapter 7

Solving more complicated problems directly: An invitation to Strum-Liouville theory

Consider the following IBVP:

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(t,0) = 0 \\ u_x(t,1) + hu(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}.$$

We can solve this using separation of variables. First, we seek product solutions of the form:

$$u(t,x) = T(t)X(x).$$

By asking that $u_t = \alpha^2 u_{xx}$, we obtained

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \mu$$

where μ is a constant. What are the possibilities for μ?

7.1 $\mu > 0$

If $\mu > 0$, we obtain

$$u(t,x) = Ae^{\alpha^2 \mu t} X(x).$$

But as $e^{\alpha^2 \mu t} \to \infty$ as $t \to \infty$, we reject this solution on physical grounds.

7.2 $\mu = 0$

If $\mu = 0$, then $T'(t) = X''(x) = 0$, so

$$u(t,x) = Ax + b.$$

To satisfy the BCs in this case:

$$u(t,0) = 0 = A \times 0 + b.$$

So, $b = 0$. Next,

$$u_x(t,1) + hu(t,1) = 0.$$

So, $A + hA = 0$, so $A = 0$. This is just the trivial solution.

7.3 $\mu < 0$

Let $\mu = \lambda^2$, then we have

$$T(t) = Ae^{-(\alpha\lambda)^2 t}$$

and

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x).$$

So,

$$u(t,x) = e^{-(\lambda\alpha)^2 t}\left(A\sin(\lambda x) + B\cos(\lambda x)\right).$$

We let this subject to BCs:

$$u(t,0) = e^{-(\lambda\alpha)^2 t}\left(A\sin(\lambda \times 0) + B\cos(\lambda \times 0)\right) = 0.$$

So, $B = 0$. Thus our product solution looks like

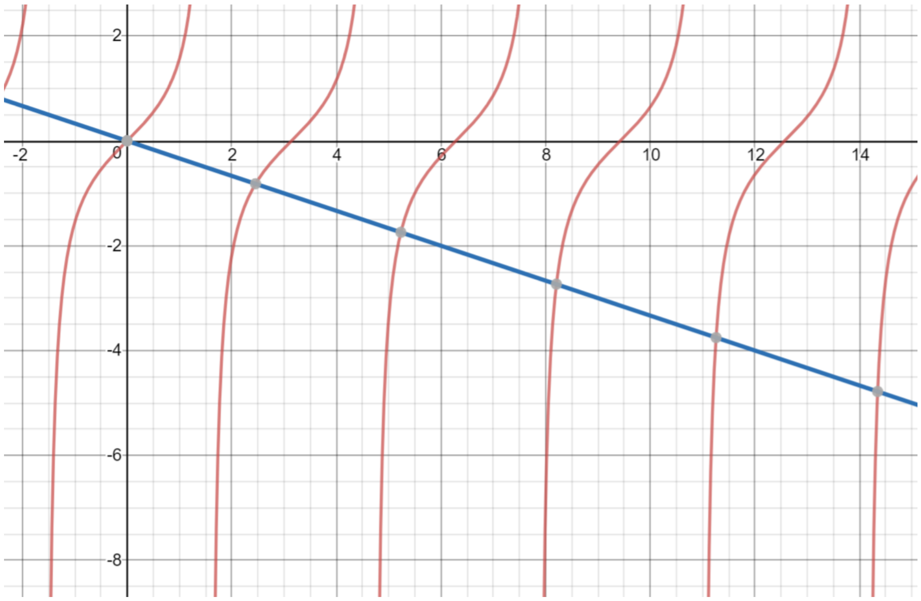
$$u(t,x) = Ae^{-(\lambda\alpha)^2 t}\sin(\lambda x).$$

The other BC gives

$$0 = u(t,1) + hu_x(t,1) = Ae^{-(\lambda\alpha)^2 t}\left(\lambda\cos(\lambda) + h\sin(\lambda)\right).$$

So,

$$\tan\lambda - \frac{-\lambda}{h}.$$



So, admissible λ's here are solution to this equation, which is not as simple as πn, like we have found before. Now, note that λ is this equation cannot be found explicitly. Solutions are just intersections on the following plot:

We see that

$$\frac{\pi}{2} < \lambda_1 < \pi < \lambda_2 < 2pi < \lambda_3 < 3\pi.$$

We call these λ_n's eigenvalues associated with the boundary value problem

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X(0) = 0 \\ X'(1) + hX(1) = 0 \end{cases} . \tag{7.1}$$

The solutions, $\sin(\lambda_n x)$ are associated eigenfunction. So, with separation of variables, we obtain

$$u_n(x,t) = A_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x)$$

which solve the IBVP. Once again, we have that

$$u(t,x) = \sum_{n=1}^{\infty} A_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x)$$

which we want to also satisfy the IC, which is

$$u(0,x) = u_0(x).$$

We invoke Fourier's trick to find $A_n's$:

$$\begin{aligned}\int_0^1 u_0(x)\sin(\lambda_m x)\,dx &= \int_0^1 \sin(\lambda_m x)\sum_{n=1}^{\infty} A_n e^{-(\lambda_n\alpha)^2 t}\sin(\lambda_n x)\,dx\\ &= \sum_{n=1}^{\infty} A_n \int_0^1 \sin(\lambda_m x)\sin(\lambda_n x)\,dx\\ &= A_m\left(\frac{1}{2}-\frac{\sin(2\lambda_m x)}{4\lambda_m}\right).\end{aligned}$$

So

$$\int_0^1 u_0(x)\sin(\lambda_m x)\,dx = A_m\frac{1}{2\lambda_m}\left(\lambda_m - \sin(\lambda_m x)\cos(\lambda_m x)\right)$$

and so

$$A_m = \frac{2\lambda_m}{\lambda_m - \sin(\lambda_m x)\cos(\lambda_m x)}\int_0^1 u_0(x)\sin(\lambda_m x)\,dx.$$

Summary: Recall the IBVP:

$$\begin{cases} u_t = \alpha^2 u_{xx}\\ u(t,0)=0\\ u_x(t,1)+hu(t,1)=0\\ u(0,x)=u_0(x)\end{cases}.$$

In trying to solve it, we have found

$$u(t,x)\sum_{n=1}^{\infty} A_n e^{-(\alpha\lambda)^2 t}\sin(\lambda_n x)$$

to be a solution with

$$A_m = \frac{2\lambda_m}{\lambda_m - \sin(\lambda_m x)\cos(\lambda_m x)}\int_0^1 u_0(x)\sin(\lambda_m x)\,dx.$$

where λ_n is the nth positive solution to

$$\frac{-\lambda}{n} = \tan(\lambda)$$

Question: In our finding of A_n, we swept a detail under the rug. It was **orthogonality**. We assumed the fact that for all n, m

$$\int_0^1 \sin(\lambda_m x)\sin(\lambda_n x)\,dx = 0, \qquad \text{if } m \neq n.$$

This would be obvious if $\lambda_n = n\pi$. But it is not the case here. We will look at this in the next section.

Chapter 8

ODE Boundary Value Problems - a look at the Sturm-Liouville theory

(reference: Boyce-DiPrima, Chapter 11).

Motivating example:

$$\begin{cases} ODE: y'' + \lambda y = 0 \\ BC: y(0) = y(1) = 0, 0 \leq x \leq 1. \end{cases}.$$

We ask: does solving this BVP require anything about λ? Is it possible to find non-zero solutions to this problem for arbitrary λ? We can try...

If $\lambda = 0$, then $y(x) = Ax + B$. By subjecting to the BCs, we get $A = B = 0$. So, not every λ gives a non-trivial solution. This λ does not admit non-zero solutions. This is in stark contrast to IVP which can be solved for any non-zero λ.

If $\lambda < 0$, we get the same issue (can check that there are no non-zero solutions for $\lambda < 0$.

If $\lambda > 0$, then $y(x) = C_1 \sin\left(\sqrt{\lambda}x\right) + C_2 \cos\left(\sqrt{\lambda}x\right)$. Subject to boundary conditions, we have $C_2 = 0$ and

$$C_1 = \sin\left(\sqrt{\lambda}x\right).$$

For non-trivial solutions,

$$\sqrt{\lambda} = n\pi.$$

Or for $n = 1, 2, 3, \ldots$

$$\lambda_n = n^2\pi^2.$$

The moral: Not every λ works. So we ask: Given a linear ODE and linear homogeneous BCs, which λ (if any) will work?

Definition 8.0.1. General theory: Consider the ODE

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0,$$

where we will let $p, q, r \in C^0([0,1])$. Moreover, $p(x) \in C^1([0,1])$. Finally, $p(x), r(x) > 0$ for all $x \in [0,1]$. Also, consider the linear homogeneous BCs:

$$\begin{aligned} a_1 y(0) + b_1 y'(0) &= 0 \\ a_2 y(1) + b_2 y'(1) &= 0. \end{aligned}$$

If, for some fixed λ, the ODE and the BCs have a non-zero solution $\phi(x)$, we say that λ is an **eigenvalue** for the BVP, and $\phi(x)$ is an **eigenfunction** corresponding to λ.

An eigenvalue λ corresponding to the BVP is said to be simple if it does not have two linearly independent eigenfunctions ϕ_1, ϕ_2. □

Remark 8.0.1.

1. We will write

   $$L[y] = -(p(x)y')' + q(x)y,$$

   which is a second-order linear operator. Then the ODE becomes:

   $$L[y] = \lambda r(x)y.$$

   For this reason, we see the "eigen" terminology arises.

Given our ODE, the theory of ODE gives, for fixed λ, two linearly independent solutions of the form $y_1 = y_1(x, \lambda), y_2 = y_2(x, \lambda)$, and so all solutions to the ODE is of the form

$$y(x) = C_1 y_1(x, \lambda) + C_2 y_2(x, \lambda).$$

Question: can we find a condition on λ so that y above is a non-zero solution satisfying the BCs? Sure we can. Let us subject y to the BCs. We have

$$\begin{aligned} 0 = a_1 y(0) + b_1 y'(0) =& a_1(C_1 y_1(0,\lambda) + C_2 y_2(0,\lambda)) + b_1(C_1 y_1'(0,\lambda) + C_2 y_2'(0,\lambda)) \\ &= C_1(a_1 y_1(0,\lambda) + b_1 y_1'(0,\lambda)) + C_2(a_1 y_2(0,\lambda) + b_1 y_2'(0,\lambda)). \end{aligned}$$

Similarly,

$$0 = C_1(a_2 y_1(1,\lambda) + b_2 y_1'(1,\lambda)) + C_2(a_2 y_2(1,\lambda) + b_2 y_2'(1,\lambda)).$$

As a matrix equation,

$$A\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 y_1(0,\lambda) + b_1 y_1'(0,\lambda) & a_1 y_2(0,\lambda) + b_1 y_2'(0,\lambda) \\ a_2 y_1(1,\lambda) + b_2 y_1'(1,\lambda) & a_2 y_2(1,\lambda) + b_2 y_2'(1,\lambda) \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

To get non-trivial solution, we require that $\ker(A) \neq \{0\}$, i.e., $\det(A) = 0$. This gives a necessary and sufficient condition on λ, since a_i, b_i and the solutions are known, that is

$$\boxed{\det(A) = 0}$$

Recall that

$$L[y] = -(p(x)y')' + q(x)y.$$

So the ODE is

$$L[y] = \lambda r(x)y.$$

Proposition 8.0.1. Lagrange's Identity: For any $u, v \in \mathbb{C}^2([0,1])$, the following identity holds:

$$\int_0^1 (uL[v] - vL[u])\,dx = p(x)\left(u'(x)v(x) - u(x)v'(x)\right)\Big|_0^1.$$

Intuitively, this is essentially an integration by parts formula for the operator L.

Proof. We have

$$\begin{aligned} u(x)L[v](x) - v(x)L[u](x) &= u(x)(-(p(x)v')' + q(x)v) - v(x)(-(p(x)u')' + q(x)u) \\ &= -u(x)(p(x)v'(x))' + u(x)q(x)v(x) \\ &\quad + v(x)(p(x)u(x))' - q(x)v(x)u(x) \\ &= -u(x)(p(x)v'(x))' + v(x)(p(x)u(x))'. \end{aligned}$$

So the integral becomes

$$\begin{aligned} \int_0^1 (uL[v] - vL[u])\,dx &= \int_0^1 -u(x)(p(x)v'(x))' + v(x)(p(x)u(x))'\,dx \\ &= \int_0^1 -u(x)(p(x)v'(x))'\,dx + \int_0^1 v(x)(p(x)u(x))'\,dx \\ &= -u(x)(p(x)v'(x))\Big|_0^1 - \int_0^1 -u'(x)p(x)v'(x)\,dx \\ &\quad + v(x)(p(x)u'(x))\Big|_0^1 - \int_0^1 v'(x)(p(x)u'(x))\,dx \\ &= p(x)\left(v(x)u'(x) - u(x)v'(x)\right)\Big|_0^1. \end{aligned}$$

□

Corollary: If u, v are eigenfunctions for the BVP then

$$\int_0^1 (uL[v] - vL[u])\, dx = 0.$$

Proof. Recall the boundary conditions:

$$\begin{aligned} a_1 u(0) + b_1 u'(0) &= 0 \\ a_1 v(0) + b_1 v'(0) &= 0 \\ a_2 u(1) + b_2 u'(1) &= 0 \\ a_2 v(1) + b_2 v'(1) &= 0 \end{aligned}$$

Assume that $b_1, b_2 \neq 0$. For $\phi = v$ or u,

$$\begin{aligned} \phi'(0) &= \frac{-a_1}{b_1}\phi(0) \\ \phi'(0) &= \frac{-a_2}{b_2}\phi(1). \end{aligned}$$

So

$$\begin{aligned} \int_0^1 (uL[v] - vL[u])\, dx &= p(x)\left(v(x)u'(x) - u(x)v'(x)\right)\Big|_0^1 \\ &= p(1)\left(v(1)u'(1) - u(1)v'(1)\right) - p(0)\left(v(0)u'(0) - u(0)v'(0)\right) \\ &= \ldots \\ &= 0. \end{aligned}$$

□

Theorem: All eigenvalues associated with the Sturm-Liouville problem are real.

Proof. Let $\lambda = u + iv$ and $\phi = U + iV$ are an eigenvalue-eigenfunction pair associated with the BVP. Then

$$L[\phi] = \lambda r(x)\phi.$$

Conjugating and noting that p, q, r are real give

$$\begin{aligned} L[\phi]^* &= (\lambda r(x)\phi)^* \\ &= ((-p(x)\phi'(x))' + q(x)\phi(x))^* \\ &= (-p(x){\phi^*}')' + q(x)\phi^* \\ &= L[\phi^*]. \end{aligned}$$

This gives

$$L[\phi^*] = \lambda^* r(x)\phi^*.$$

This says that λ^* is an eigenvalue too, which takes ϕ^* as an eigenfunction. So, (λ^*, ϕ^*) and (λ, ϕ) are two eigenvalue-eigenfunction pairs. By the corollary:

$$\begin{aligned} 0 &= \int_0^1 (\phi L[\phi^*] - \phi^* L[\phi])\, dx \\ &= \int_0^1 (\lambda^* - \lambda) r(x) |\phi(x)|^2\, dx \end{aligned}$$

Since $r(x) > 0$ for all x, and ϕ is non-zero,

$$|\phi(x)|^2 > 0$$

for some non-trivial interval in $[0, 1]$. So

$$(\lambda^* - \lambda) \int_0^1 r(x)|\phi(x)|^2\, dx > 0.$$

And so

$$\lambda^* = \lambda,$$

i.e., $\lambda \in \mathbb{R}$. □

Theorem: ORTHOGONALITY - Sweeping under the carpet: Let λ_1 and λ_2 be distinct eigenvalues with eigenfunctions ϕ_1 and ϕ_2. Then

$$\int_0^1 r(x)\phi_1(x)\phi_2(x)\, dx = 0.$$

Proof. We have, by the corollary, that

$$\begin{aligned} 0 &= \int_0^1 (\phi_1 L[\phi_2] - \phi_2 L[\phi_1])\, dx \\ &= \int_0^1 r(x)\phi_1\lambda_2\phi_2 - \phi_2\lambda_1 r(x)\phi_1\, dx \\ &= (\lambda_2 - \lambda_1) \int_0^1 r(x)\phi_1\phi_2\, dx. \end{aligned}$$

Since the eigenvalues are distinct, the integral must be zero. □

So, if ϕ_1 and ϕ_2 were eigenfunctions corresponding to distinct eigenvalues λ_1, λ_2 then ϕ_1 and ϕ_2 are orthogonal in the sense given by the theorem.

But how do we know ϕ's exist?

Theorem: Given a Sturm-Liouville problem, there exist eigenvalues and eigenfunctions. The collection of eigenvalues forms an infinite sequence $\{\lambda_n\}$ for which $\lambda_1 < \lambda_2 < \dots$ and $\lambda_n \to \infty$ as $n \to \infty$, Further, each eigenvalue is simple. (Note: this has to do with the spectral theorem).

The final result is another important theorem. But first we define a few things.

Definition 8.0.2. An eigenfunction for the Sturm-Liouville problem is said to be normalized if

$$\int_0^1 \phi^2 r(x)\, dx = 1.$$

Definition 8.0.3. In view of the Sturm-Liouville problem, a function f is admissible with respect to the S-L problem if it is continuous, piecewise differentiable on $[0,1]$ and

1. If $b_1 = 0$ then $f(0) = 0$.
2. If $b_2 = 0$ then $f(1) = 0$.

Theorem: Completeness. Let $\{\phi_n\}$ be the sequence of normalized eigenfunctions to the S-L problem.

1. If f is admissible with respect to the S-L problem, then $f(x)$ can be expressed as

$$\sum_{i=1}^{\infty} a_i \phi_i(x)\, dx$$

where

$$a_i = \int_0^1 f(x)\phi_i(x) r(x)\, dx.$$

and $f(x)$ converges pointwise.

Example 8.0.1. Consider a S-L problem:

$$u'' + \lambda u = 0$$
$$\begin{cases} u(0) = 0 & (a_1 = 1, b_1 = 0) \\ u(1) = 0 & (a_2 = 1, b_2 = 0) \end{cases}$$

We found that

$$\lambda_n = (n\pi)^2$$

with the eigenfunctions being $\{\sqrt{2}\sin(n\pi x)\}$. The theorem says if f is a continuous, piecewise differentiable functions on the interval $[0,1]$vwith $f(0) = 0 = f(1)$, then

$$f(x) = \sum_{n=1}^{\infty} a_n \left(\sqrt{2}\sin(n\pi x)\right) \quad \text{(Fourier series)}$$

where

$$a_n = \int_0^1 f(x)\left(\sqrt{2}\sin(n\pi x)\right)\, dx$$

Observe that for the S-L problem we have the linear operator $L : u \to L[u]$:

$$L[u] = -(p(x)u')' + q(x)u.$$

L is a linear operator initially defined on $C^2([0,1])$ (twice-differentiable functions on) and $L : C^2([0,1]) \to C^0([0,1])$. So, given boundary conditions for the problem of the form

$$\begin{aligned} a_1 u(0) + b_1 u'(0) = 0 \\ a_2 u(1) + b_2 u'(1) = 0, \end{aligned}$$

any twice differentiable functions u, v which satisfy the boundary conditions have the following property that

$$\int_0^1 uL[v] - vL[u]\, dx = 0.$$

If, say

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\, dx$$

then

$$\langle u, L[v]\rangle = \langle L[u], v\rangle.$$

Such an operator L is said to be (formally) self-adjoint.

Chapter 9

Transforming Hard Equations into Easier Ones

In this section we will learn how to solve the IBVP of a laterally-heat-losing rod.

$$u_t = \alpha^2 u_{xx} - \beta u$$
$$\begin{cases} u(t,0) = 0 = u(t,1) \\ u(0,x) = u_0(x) \end{cases}$$

Ansatz: we will assume (so that the β term goes away)

$$u(t,x) = e^{-\beta t} w(t,x).$$

Can we convert an equation for u into an easier equation for w. Plug in, we get:

$$-\beta e^{\beta t} w(t,x) + e^{-\beta t} w_t(t,x) = \alpha^2 e^{-\beta t} w''(t,x) - \beta e^{-\beta t} w(t,x).$$

This gives a nice equation for w:

$$w_t = \alpha^2 w_{xx}.$$

A moment's thougt shows that all of these steps are reversible. This means that w solves $w_t = \alpha^2 w_{xx} \iff u$ solves $u_t = \alpha^2 u - \beta u$.

The boundary and initial conditions are exactly the same. In fact, u satisfies the BCs and ICs $\iff$ w satisfies the same ones:

$$\begin{cases} w(t,0) = 0 = w(t,1) \\ w(0,x) = u(0,x) = u_0(x). \end{cases}$$

So, we wish to solve the following problem:

$$w_t = \alpha^2 w_{xx}$$
$$\begin{cases} w(t,0) = 0 = w(t,1) \\ w(0,x) = u_0(x) \end{cases} .$$

The solution is of course

$$w(t,x) = \sum_{i=1}^{\infty} a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)$$

where

$$a_n = 2\int_0^1 u_0(x) \sin(n\pi x)\, dx.$$

So

$$\begin{aligned} u(t,x) &= e^{-\beta x} w(t,x) \\ &= e^{-\beta t} \sum_{i=1}^{\infty} a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \\ &= \sum_{i=1}^{\infty} a_n e^{-[(n\pi\alpha)^2 + \beta]t} \sin(n\pi x). \end{aligned}$$

Since $\beta > 0$, $u(t,x) \to 0$ faster than if $\beta = 0$.

Chapter 10

Solving Nonhomogeneous PDEs (Eigenfunction Expansion)

In all we've done, we haven't yet attack an inhomogeneous heat equation. Let's set out to solve the problem

$$u_t = \alpha^2 u_{xx} + f(t,x)$$
$$\begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} \quad u(0,x) = u_0(x).$$

The method we shall follow will also solve the more general problem

$$u_t = \alpha^2 u_{xx} + f(t,x)$$
$$\begin{cases} a_1 u(t,0) + b_1 u_x(t,0) = 0 \\ a_2 u(t,1) + b_2 u_x(t,1) = 0 \end{cases} \quad u(0,x) = u_0(x).$$

The question is: Can we write the solution as a series of products:

$$u(t,x) = \sum_{n=0}^{\infty} T_n(t) X_n(x)?$$

Recall that we've expanded things via Fourier series (generally to deal with u_0) or general eigenfunction expansions from the Sturm-Liouville theory. So, can we also expand $f(t,x)$ in this way?

Consider the (easy) inhomogeneous problem.

$$u_t = \alpha^2 u_{xx} + f(t,x)$$
$$\begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} \quad u(0,x) = u_0(x).$$

We will solve with via eigenfunction expansion...

1. Decompose f into pieces so that

$$f(t,x) = \sum_{n=1}^{\infty} f_n(t) X_n(x)$$

where X_n's arise by solving the associated homogeneous problem:

$$u_t = \alpha^2 u_{xx}$$
$$\begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} \quad u(0,x) = u_0(x).$$

To this end, we write the solution

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

where X_n's solve the Strum-Liouville problem:

$$X'' - \alpha^2 X_{xx} = 0$$
$$\begin{cases} X(0) = 0 \\ X(1) = 0 \end{cases}$$

For this particular problem,

$$X_n(x) = \sin(n\pi x).$$

With the assumption that the function $X \to f(t,x)$ is admissible in the Sturm-Liouville sense for all t, the result from $S - L$ theory gives that

$$f(t,x) = \sum_{n=1}^{\infty} f_n(t) X_n(x),$$

where

$$f_n(t) = 2\int_0^1 f(t,x)\sin(n\pi x)\,dx.$$

2. Subject

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

to the IBVP, where

$$f(t,x) = \sum_{n=1}^{\infty} f_n(t) X_n(x),$$

and

$$X_n(x) = \sin(n\pi x),$$

we get

$$u_t(t,x) = \sum_{n=1}^{\infty} \dot{T}_n(t)\sin(n\pi x),$$

and

$$u_{xx}(t,x) = \sum_{n=1}^{\infty} -(n\pi)^2 T_n(t)\sin(n\pi x).$$

Into the IBVP, $u_t = \alpha^2 u_{xx}$

$$\sum_{n=1}^{\infty} \dot{T}_n(t)\sin(n\pi x) = \alpha^2 \sum_{n=1}^{\infty} -(n\pi)^2 T_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x),$$

or equivalently,

$$\sum_{n=1}^{\infty} \left[\dot{T}_n(t) + (\alpha n\pi)^2 T_n(t)\right]\sin(n\pi x) = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x).$$

This yields ODEs of the form:

$$\dot{T}_n(t) + (\alpha n\pi)^2 T_n(t) = f_n(t), \qquad n = 1, 2, 3, \ldots$$

to which the solution is of the form

$$T_n(t) = e^{-(\alpha n\pi)^2 t}\int_0^t f_n(s)e^{(\alpha n\pi)^2 s}\,ds + T_n(0)e^{-(\alpha n\pi)^2 t}.$$

Note, for

$$u(0,x) = \int_{n=1}^{\infty} T_n(0)\sin(n\pi x) = u_0(x),$$

which means the choice of the Fourier Sine coefficients

$$T_n(0) = 2\int_0^1 u_0(x)\sin(n\pi x)\,dx, \qquad n = 1, 2, 3, \ldots$$

does the trick. Thus, the original IBVP is satisfied $^{(?)}$ with

$$u(t,x) = \sum_{n=1}^{\infty}\left(e^{-(\alpha n\pi)^2 t}\int_0^t f_n(s)e^{(\alpha n\pi)^2 s}\,ds + T_n(0)e^{-(\alpha n\pi)^2 t}\right)\sin(n\pi x),$$

where

$$T_n(0) = 2\int_0^1 u_0(x)\sin(n\pi x)\,dx, \qquad n = 1, 2, 3, \ldots$$
$$f_n(t) = 2\int_0^1 f(t,x)\sin(n\pi x)\,dx, \qquad n = 1, 2, 3, \ldots$$

We write this as

$$u(t,x) = \sum_{n=1}^{\infty}\left(e^{-(\alpha n\pi)^2 t}\int_0^t f_n(s)e^{(\alpha n\pi)^2 s}\,ds\right)\sin(n\pi x) + \sum_{n=1}^{\infty} T_n(0)e^{-(n\pi\alpha)^2 t}\sin(n\pi x),$$

where the first term is the steady-state solution, and the second term is the transient solution, because in the second term, as $t \to \infty$ it goes to zero. Let's check that this works.

(a) Note that $u(t,0) = u(t,1) = 0$.

(b) And

$$u(0,x) = 0 + \sum_{n=1}^{\infty} T_n(0)e^{-(n\pi\alpha)^2 t}\sin(n\pi x) = u_0(x),$$

which is true by design.

(c) We can also check that it solves the PDE, but we won't...

Example 10.0.1.

$$\begin{cases} u_t = u_{xx} + x - x^2 \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) = \sin(\pi x). \end{cases}$$

Solution. Here, $f(t,x) = x - x^2$. So

$$
\begin{aligned}
f_n(t) &= 2\int_0^1 (x - x^2)\sin(n\pi x)\,dx \\
&= 2\left((x - x^2)\frac{\cos(n\pi x)}{-n\pi}\bigg|_0^1 - \int_0^1 (1-2x)\frac{\cos(n\pi x)}{-n\pi}\,dx \right) \\
&= 0 + \frac{2}{n\pi}\int_0^1 (1-2x)\cos(n\pi x)\,dx \\
&= \frac{2}{n\pi}\left((1-2x)\frac{\sin(n\pi x)}{n\pi}\bigg|_0^1 - \frac{2}{n\pi}\int_0^1 (-2)\frac{\sin(n\pi x)}{n\pi}\,dx \right) \\
&= \frac{4}{(n\pi)^2}\int_0^1 \sin(n\pi x)\,dx \\
&= \frac{4}{(n\pi)^3}(-\cos(n\pi x))\bigg|_0^1 \\
&= \frac{4}{(n\pi)^3}(1-(-1)^n).
\end{aligned}
$$

Also,

$$T_n(0) = \int_0^1 u_0(x)\sin(n\pi x)\,dx,$$

but since $u_0(x) = \sin(\pi x)$,

$$T_n(0) = \delta_n^1.$$

Now,

$$
\begin{aligned}
\int_0^t f_n(s)e^{(n\pi)^2 s}\,ds &= \int_0^t \frac{4}{(n\pi)^3}(1-(-1))^n e^{(n\pi)^2 s}\,ds \\
&= \frac{4}{(n\pi)^5}(1-(-1)^n)\left(e^{(n\pi)^2 t} - 1\right).
\end{aligned}
$$

All together...

$$u(t,x) = \sum_{n=1}^{\infty}\left(e^{-(n\pi)^2 t}\left(\frac{4}{(n\pi)^5}(1-(-1)^n)\left(e^{(n\pi)^2 t} - 1\right)\right)\right)\sin(n\pi x) + e^{-(\pi)^2 t}\sin(\pi x).$$

Physically, what does this solution say? First, we have a diffusion term. But not only that, we have $u_t \sim x - x^2$, i.e., we're adding heat to the middle of the rod and allowing this heat to diffuse. We can also approximate the solution to get

$$
\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty}\left(e^{-(n\pi)^2 t}\left(\frac{4}{(n\pi)^5}(1-(-1)^n)\left(e^{(n\pi)^2 t} - 1\right)\right)\right)\sin(n\pi x) + e^{-(\pi)^2 t}\sin(\pi x) \\
&\approx \frac{8}{\pi^5}\sin(\pi x) + \left(1 - \frac{8}{\pi^5}\right)e^{-\pi^2 t}\sin(\pi x).
\end{aligned}
$$

So this makes sense, because the temperature distribution is $\sin(\pi x)$ scaled down by some number.

Chapter 11

Integral Transformation

An integral transformation/operator is a map taking a function f to another function $F = \mathcal{I}[f]$ by the rule:

$$\boxed{\mathcal{I}[f](s) = \int_A^B \mathcal{K}(s,t)f(t)\,dt}$$

where $\mathcal{K}(s,t)$ is called an **integral kernel**.

Proposition 11.0.1. Integral operators are linear (when taken to be defined on some appropriate vector space of functions).

Proof. -ish

Let f, g be given. Then

$$\begin{aligned}
\mathcal{I}[\alpha f + \beta g](s) &= \int_A^B \mathcal{K}(s,t)\left(\alpha f(t) + \beta g(t)\right)\,dt \\
&= \alpha \int_A^B \mathcal{K}(s,t)f(t)\,dt + \beta \int_A^B \mathcal{K}(s,t)g(t)\,dt \\
&= \alpha\mathcal{I}[f](s) + \beta\mathcal{I}[g](s).
\end{aligned}$$

□

Note: The study of integral transformations is a main focus of functional analysis.

Note: We can think of this as moving between spaces: momentum and position.

In many cases, an integral transformation $\mathcal{I}$ will have an inverse, denoted $\mathcal{I}^{-1}$, so that, in particular, if $F(s) = \mathcal{I}[f](s)$ then $f(t) = \mathcal{I}^{-1}[F](t)$, i.e.,

$$\mathcal{I}^{-1} \circ \mathcal{I} = \text{Identity}.$$

For us, presently, we will work with integral transformations to take difficult PDEs and transform them into simple PDEs which are easier to solve . Taking the solution to the easier PDE and applying the inverse transformation will yield the solution to the harder PDE.

Idea: Hard PDE $\to$ Easier PDE $\to$ Solution to easy PDE $\to$ Solution to hard PDE.

Note: there's some resemblance between this and "change of basis."

11.1 Some common transformations

11.1.1 The Fourier Transform

Works with $f : \mathbb{R} \to \mathbb{C}$.

$$\mathcal{F}[t](\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx = F(\xi).$$

The inverse looks almost exactly the same:

$$\mathcal{F}^{-1}[F](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} F(\xi)\, d\xi = f(x).$$

This yields (in the vector space $L^2(\mathbb{R})$)

$$\mathcal{F}^{-1} \circ \mathcal{F} = \text{Identity}.$$

Disadvantage: complex-valued things

Advantage: almost always works, and takes differentiation to polynomial multiplication.

Note: $\mathcal{F}$ is a unitary operator.

11.1.2 The Fourier Sine Transform

$$\mathcal{F}_s[f](w) = \frac{2}{\pi} \int_0^{\infty} \sin(wt) f(t)\, dt.$$

The inverse is

$$\mathcal{F}_s^{-1}[F](t) = \int_0^{\infty} \sin(wt) F(w)\, dw.$$

11.1.3 The Fourier Cosine Transform

$$\mathcal{F}_c[f](w) = \frac{2}{\pi} \int_0^\infty \cos(wt) f(t)\, dt.$$

The inverse is

$$\mathcal{F}_c^{-1}[F](t) = \int_0^\infty \cos(wt) F(w)\, dw.$$

11.1.4 The Discrete/Finite Fourier Transform (Fourier Series)

Given a function $f : [0, L] \to \mathbb{R}$ or $\mathbb{C}$, the finite Fourier transform

$$\mathcal{F}[f](n) = \hat{f}(n) = a_n = \frac{1}{L} \int_0^L f(x) e^{-2\pi i n x/L}\, dx.$$

The inverse is

$$\mathcal{F}^{-1}(n) = \mathcal{F}^{-1}[\hat{f}(n)] = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x/L}.$$

Property:

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} = \mathcal{F}^{-1} \circ \mathcal{F}[f].$$

11.1.5 The Analogous Sine Transform

$$\mathcal{F}_s[f](n) = a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx.$$

The inverse is

$$\mathcal{F}_s^{-1}[a_n](x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right).$$

Disadvantage: Half-range expansions...

11.1.6 The Laplace Transform

Note: the Laplace transform is analogous to the Moment Generating Function in Probability theory.

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt.$$

The (difficult) inverse is

$$\mathcal{L}^{-1}[F](t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} F(s)e^{st}\,ds,$$

which is a complex contour integral.

11.2 The Fourier Series

We had:

$$\begin{aligned}\mathcal{F}_s[f](\omega) &= \frac{2}{\pi}\int_0^1 f(x)\sin(\omega x)\,dx\\ \mathcal{F}_c[f](\omega) &= \frac{2}{\pi}\int_0^1 f(x)\cos(\omega x)\,dx\\ \mathcal{F}[f](\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx.\end{aligned}$$

Proposition 11.2.1. Some identities:

1. $\mathcal{F}_s[f'] = -\omega\mathcal{F}_c[f]$
2. $\mathcal{F}_s[f''] = \frac{2}{\pi}\omega f(0) - \omega^2\mathcal{F}_s[f]$
3. $\mathcal{F}_c[f'] = \frac{-2}{\pi}f(0) + \omega\mathcal{F}_s[f]$
4. $\mathcal{F}_c[f''] = \frac{-2}{\pi}f'(0) - \omega^2\mathcal{F}_c[f]$.
5. $\mathcal{F}[f'](\xi) = i\xi\mathcal{F}[f]$
6. $F[f''](\xi) = -\xi^2\mathcal{F}[f](\xi)$

Remark 11.2.1. In our alternative notation, this says

$$\hat{f}'(\xi) = i\xi\hat{f}(\xi).$$

We notice that the $\hat{}$ map converts differentiation to polynomial multiplication.

Proof of propositions 5,6. 1. Proof of 5.: The supposition that f, f' have existent Fourier Transforms, i.e.,

$$\int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx, \int_{-\infty}^{\infty} f'(x)e^{-ix\xi}\,dx$$

exist for all ξ, requires in particular, that

$$\lim_{t\to\pm\infty} f(x) = \lim_{t\to\pm\infty} f'(x) = 0.$$

So, interpreting these as improper Riemann integrals:

$$\begin{aligned}
\mathcal{F}[f'](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \lim_{t\to\infty} \int_{-t}^{t} f'(x) e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \lim_{t\to\infty} \left(f(x)e^{-ix\xi}\Big|_{-t}^{t} - \int_{-t}^{t} f(x)(-i\xi)e^{-ix\xi}\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \lim_{t\to\infty} \left(f(t)e^{0it\xi} - f(-t)e^{it\xi} - i\xi \int_{-t}^{t} f(x)e^{-ix\xi}\, dx \right) \\
&= i\xi\mathcal{F}[f](\xi).
\end{aligned}$$

2. Proof of 6.:

$$\begin{aligned}
\mathcal{F}[f'](\xi) &= i\xi\mathcal{F}[f'](s) \\
&= (i\xi)^2\mathcal{F}[f](\xi) \\
&= -\xi^2\mathcal{F}[f](\xi).
\end{aligned}$$

□

There's a **full version** of the Fourier transform that we won't discuss here.

Rather, let us use the $\mathcal{F}_s$ transform to solve half-infinite rod heat equation:

$$\begin{cases} u_t = \alpha^2 u_{xx}, & 0 < x < \infty, t > 0 \\ u(t,0) = A, & t > 0 \\ u(0,x) = 0, & x \geq 0. \end{cases}$$

We start with $u_t = \alpha^2 u_{xx}$. Let's compute the Fourier Sine transform in the variable x (this is sometimes called the **Partial Fourier Transform**):

$$\begin{aligned}
\mathcal{F}[u_t(t,x)](\omega) &= \frac{2}{\pi} \int_0^{\infty} u_t(t,x) \sin(\omega x)\, dx \\
&= \frac{2}{\pi} \frac{\partial}{\partial t} \left(\int_0^{\infty} u(t,x) \sin(\omega x)\, dx \right) \\
&= \frac{\partial}{\partial t} \mathcal{F}_s[u(t,x)](\omega).
\end{aligned}$$

We shall call

$$U(t,\omega) = \mathcal{F}_s[u(t,x)](\omega).$$

So,

$$\mathcal{F}_s[u_t(t,x)](\omega) = \frac{\partial U}{\partial t}.$$

On the other hand,

$$\begin{aligned}\mathcal{F}_s[u_{xx}](\omega) &= \frac{2}{\pi}\int_0^\infty u_{xx}\sin(\omega x)\,dx \\ &= \frac{2}{\pi}\omega u(t,0) - \omega^2\mathcal{F}_s[u(t,x)](\omega).\end{aligned}$$

But $u(t,0) = A$ by the boundary condition, so

$$\mathcal{F}_s[u_{xx}](\omega) = \frac{2A}{\pi}\omega - U(t,\omega).$$

Subject to $u_t = \alpha^2 u_{xx}$, we claim:

$$\frac{\partial U}{\partial t} = \alpha^2\left(\frac{2A\omega}{\pi} - \omega^2 U(t,\omega)\right),$$

or equivalently,

$$\frac{\partial U}{\partial t} + \alpha^2\omega^2 U = \frac{2A\omega^2\alpha^2}{\pi},$$

which is nothing but a linear ODE. Now, can we get an initial condition for this ODE? Yes! Since $u(0,x) = 0$ for all x,

$$U(0,\omega) = \mathcal{F}_s[u(0,x)](\omega) = \mathcal{F}_s[0](\omega) = 0.$$

Thus, we have transformed an initial condition for our PDE in u into an initial condition in U. We get an IVP:

$$\begin{cases}\frac{\partial U}{\partial t} + \alpha^2\omega^2 U = \frac{2A\omega^2\alpha^2}{\pi} \\ U(0,\omega) = 0,\end{cases}$$

whose solution is

$$U(t,\omega) = e^{-\omega^2\alpha^2 t}\int_0^t \frac{2A}{\pi}\omega^2\alpha^2 e^{\omega^2\alpha^2 s}\,ds.$$

So,

$$U(t,\omega) = \frac{2A}{\pi}\left(1 - e^{-\omega^2\alpha^2 t}\right).$$

So, then

$$\begin{aligned}u(t,x) = \mathcal{F}_s^{-1}[U(t,\omega)] &= \mathcal{F}_s^{-1}\left[\frac{2A}{\pi}\left(1 - e^{-\omega^2\alpha^2 t}\right)\right](x) \\ &= A \times \operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right),\end{aligned}$$

where

$$\operatorname{erfc}(y) = \frac{2}{\sqrt{\pi}}\int_y^\infty e^{-t^2}\,dt.$$

11.3 From the Fourier Series to the Fourier Transform

First, let us find the connection between cosine-sine version of Fourier Series and $\sum()e^{inx}$. Here is a fact: if f is a periodic of period $2L$ and "nice," then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

for $n = 0, 1, 2, \ldots$. We note that "nice" is just piecewise differentiable, or weaker, $f \in L^2([0, L])$.

We want to connect this with the complex representation of Fourier Series:

$$f(x) = \sum_{n\in\mathbb{Z}} C_n e^{inx\pi/L},$$

where

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx.$$

Recall the Euler's identity:

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-\theta} &= \cos\theta - i\sin\theta. \end{aligned}$$

So,

$$\begin{aligned} C_n &= \frac{1}{2L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx - i\frac{1}{2L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{a_n - ib_n}{2}, \qquad n = 0, 1, 2, 3, \ldots \end{aligned}$$

For $n = -m$, $m = 0, 1, 2, 3, \ldots$ then

$$\begin{aligned} C_{-m} = C_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{im\pi x/L}\,dx \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx + i\frac{1}{2L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \\ &= \frac{a_m + ib_m}{2}. \end{aligned}$$

In other words, for all $n = 0, 1, 2, \ldots$

$$\begin{aligned} 2C_n &= a_n - ib_n \\ 2C_{-n} &= a_n + ib_n, \end{aligned}$$

then

$$a_n = C_n + C_{-n}$$

and

$$b_n = \frac{C_{-n} - C_n}{i} = i(C_n - C_{-n}).$$

Plugging these coefficients into the Fourier Series, we get

$$\begin{aligned} f(x) &= C_0 + \sum_{n=1}^{\infty}(C_n + C_{-n})\cos\frac{n\pi x}{L} + \sum_{n=0}^{\infty} i(C_n - C_{-n})\sin\frac{n\pi x}{L} \\ &= C_0 + \sum_{n=1}^{\infty} C_n\left(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} C_{-n}\left(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}\right) \\ &= C_0 + \sum_{n=1}^{\infty} C_n e^{in\pi x/L} + C_{-n}e^{-in\pi x/L} \\ &= \sum_{n\in\mathbb{Z}} C_n e^{in\pi x/L}, \end{aligned}$$

which is exactly the complex representation of the Fourier Series.

So, given a $2L$-periodic nice function f, we can write

$$\begin{aligned} f(x) &= \sum_{n\in\mathbb{Z}} C_n e^{in\pi x/L} \\ &= \sum_{n\in\mathbb{Z}}\left(\frac{1}{2L}\int_{-L}^{L} f(x')e^{-in\pi x'/L}\,dx'\right)e^{in\pi x/L} \\ &= \sum_{n\in\mathbb{Z}}\left(\frac{1}{2\pi}\int_{-L}^{L} f(x')e^{-in\pi x'/L}\,dx'\right)e^{in\pi x/L}\frac{\pi}{L} \\ &\approx \sum g_L\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}. \end{aligned}$$

What if we want to represent a function that is not $2L$-periodic. Suppose I have a Gaussian function (which is clearly not $2L$-periodic). What we do now is just have the Gaussian from $-L$ to L, and make f periodic in L (copy this part and paste everywhere). The idea is this: the Fourier Series will converge to the $2L$-periodization of f.

Notice that this, in particular, approximates the Gaussian function in the window $[-L, L]$. By taking $L \to \infty$, we can use this idea to capture all of f.

Now, consider the last expression above, we see that it is some sort of Riemann sum. As $L \to \infty$,

$$\sum g_L\left(\frac{n\pi x}{L}\right)\frac{\pi}{L} \to \int_{-\infty}^{\infty} g(\xi)\,d\xi,$$

where

$$g_L(\xi) = e^{i\xi x}\frac{1}{2\pi}\int_{-L}^{L} f(\quad)e^{-i(\quad)}\,dx' \to e^{i\xi x}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')e^{-ix'\xi}\,dx'.$$

So,

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} g(\xi)e^{i\xi x}\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')e^{-i\xi x'}\,dx'\right)e^{i\xi x}\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}\,d\xi \\ &= \mathcal{F}^{-1}[\hat{f}](x). \end{aligned}$$

We had, for a "nice" function $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$,

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ix\xi}\,dx,$$

for any $\xi \in \mathbb{R}$. This is the Fourier Transform of f. It has an inverse

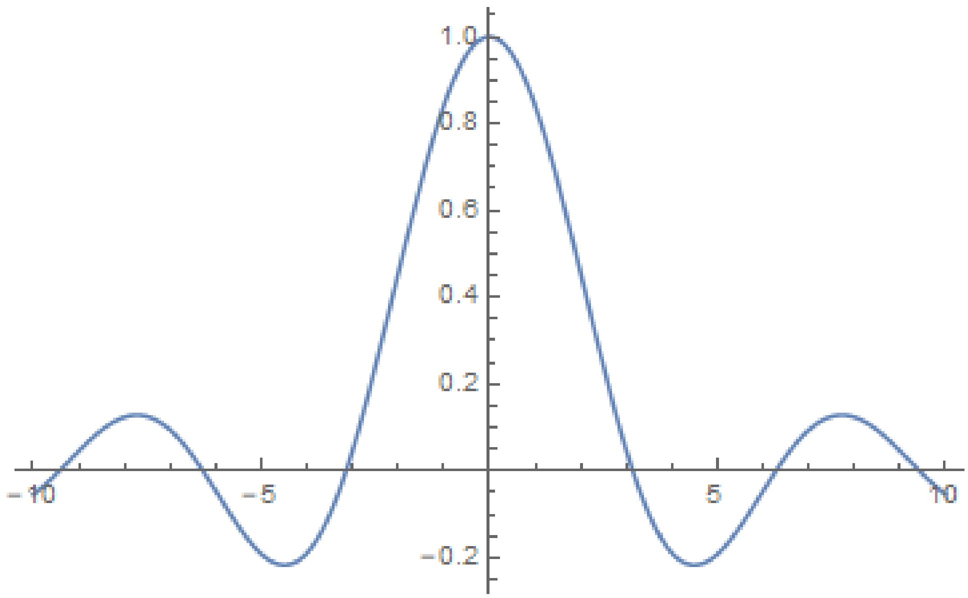
$$\mathcal{F}^{-1}[F](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\xi)e^{ix\xi}\,d\xi$$

with $\mathcal{F}^{-1} \circ \mathcal{F} = Id$.

Example 11.3.1. Let $f(x) = \nVdash_{[-1,1]}$. Then

$$\mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}}\frac{e^{-i\xi} - e^{i\xi}}{-i\xi} = \frac{2}{\pi}\frac{\sin(\xi)}{\xi}.$$

We see that the Fourier Transform takes something localized and spread it.

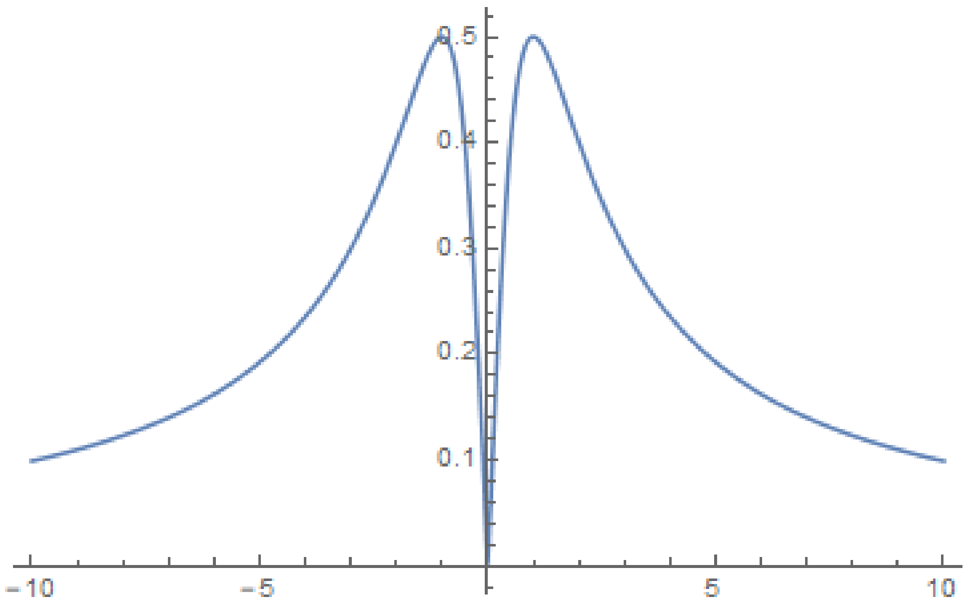


Example 11.3.2. Let

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0. \end{cases}$$

Then

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \left(\int_0^\infty e^{-(x+i\xi)}\, dx + \int_0^\infty e^{-x+i\xi x}\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_0^\infty e^{-x(1+i\xi)} - e^{-x(1-i\xi)}\, dx \right) \\
&= \frac{-1}{\sqrt{2\pi}} \left(-\frac{1}{1+i\xi} + \frac{1}{1-i\xi} \right) \\
&= \frac{1}{2\pi} \frac{-2i\xi}{1+\xi^2} \\
&= -i\sqrt{\frac{2}{\pi}} \frac{\xi}{1+\xi^2}.
\end{aligned}$$



Example 11.3.3. Consider the Gaussian function

$$f(x) = e^{-x^2}$$

then we can show

$$\hat{f}(\xi) = \frac{1}{\sqrt{2}} e^{-(\xi/2)^2}.$$

We say that the Gaussian function is in some sense the "eigenfunction" of the Fourier Transform.

Proposition 11.3.1.

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

Proof.

Definition 11.3.1.

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

So

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right).$$

So,

$$I^2 = \iint_{\mathbb{R}} e^{-(x^2+y^2)\,dxdy}$$

We change into polar coordinate

$$\begin{gathered} 0 \le r \le \infty \\ 0 \le \theta \le 2\pi. \end{gathered}$$

So

$$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\,d\theta dr = -\pi\int_{-\infty}^{\infty} 2re^{-r^2}\,dr = -\pi.$$

□

We have seen that

1. $\mathcal{F}$ is a linear map.
2. $\mathcal{F}[f'](\xi) = i\xi\mathcal{F}[f](\xi)$
3. $\mathcal{F}[f''](\xi) = -\xi^2\mathcal{F}[f](\xi)$.

But $\widehat{fg} \neq \hat{f}\hat{g}$. Though this pointwise product isn't preserved under the Fourier Transform, another product is the **Convolution Product**:

Definition 11.3.2. Given "nice" $g, f : \mathbb{R} \to \mathbb{R}$. The convolution of f and g, denoted by $f * g : \mathbb{R} \to \mathbb{R}$ defined by

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)g(x-y)\,dy.$$

Proposition 11.3.2.

$$f * g = g * f.$$

Example 11.3.4.

$$f(x) = x \qquad g(x) = e^{-x^2}.$$

Then

$$\begin{aligned} (f * g)(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2}(x-y)\,dy \\ &= \frac{x}{\sqrt{2\pi}}\sqrt{\pi} \\ &= \frac{x}{\sqrt{2}}. \end{aligned}$$

Theorem 11.3.1. For f, g "nice" enough,

$$\mathcal{F}[f * g](\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

Proof.

$$\begin{aligned}
\mathcal{F}[f * g] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(x) e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) g(x-y)\, dy \right) e^{-ix\xi}\, dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(x-y) e^{-i(x-y)\xi} e^{-iy\xi}\, dy dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} \int_{-\infty}^{\infty} g(x-y) e^{-i(x-y)\xi}\, dx dy,
\end{aligned}$$

where the last line comes from Fubini's theorem. Change of variables: $u = x - y$:

$$\mathcal{F}[f * g] = \hat{g}(\xi)\hat{f}(\xi).$$

□

Chapter 12

Application of the Fourier Transform:

Consider the PDE:

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \qquad t > 0, x \in \mathbb{R} \\ u_0(x) &= u_0(x), \qquad x \in \mathbb{R}. \end{aligned}$$

There is an implicit assumption that $u(t,x)$ is integrable on $(-\infty,\infty)$ for each t. In particular, the tails of $u(t,x)$ are small so that we can apply the Fourier Transform.

Let's apply the FT to the PDE, in the x-variable. Sometimes this is called the partial FT in the literature. So, writing $U(t,\xi) = \mathcal{F}[u(t,x)](\xi)$, we have that

$$\begin{aligned} \mathcal{F}[u_t](\xi) = \mathcal{F}_x[u_t](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_t(t,x) e^{-ix\xi}\, dx \\ &= \frac{\partial}{\partial t} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t,x) e^{-ix\xi}\, dx \\ &= \frac{\partial}{\partial t} \mathcal{F}[u](\xi) \\ &= \frac{\partial}{\partial t} U(t,\xi). \end{aligned}$$

Also, we have

$$\begin{aligned} \mathcal{F}[\alpha^2 u_{xx}(t.x)](\xi) &= \alpha^2 \mathcal{F}[u_x x(t,x)](\xi) \\ &= \alpha^2(-\xi^2)\mathcal{F}[u(t,x)](\xi), \qquad \text{from last week...} \\ &= -\alpha^2 \xi^2 U(t,\xi). \end{aligned}$$

So, together we have

$$\mathcal{F}[u_t](\xi) = \mathcal{F}[\alpha^2 u_{xx}(t.x)](\xi) \implies \frac{\partial}{\partial t} U(t,\xi) = -\alpha^2 \xi^2 U(t,\xi)$$

when $U(t,\xi) = \mathcal{F}[u(t,x)](\xi)$. Notice that this is an ODE in the variable t for each fixed ξ. We can ask, what about initial conditions? Note the following:

$$U(0,\xi) = \mathcal{F}[u(0,x)](\xi) = \mathcal{F}[u_0(x)](\xi) = \hat{u}_0(\xi).$$

So, we have converted the initial IBVP into an IVP for each fixed ξ given above. That means we wish to solve the following IVP:

$$\begin{cases} \frac{\partial}{\partial t} U(t,\xi) = -\alpha^2 \xi^2 U(t,\xi) \\ U(0,\xi) = \hat{u}_0(\xi). \end{cases} \tag{12.1}$$

The solution to this IVP, for each ξ is

$$U(t,\xi) = \hat{u}_0(\xi) e^{-\alpha^2 \xi^2 t}.$$

So now we take the inverse:

$$u(t,x) = \mathcal{F}^{-1}[U(t,\xi)](x) = \mathcal{F}^{-1}\left[\hat{u}_0(\xi) e^{-\alpha^2 \xi^2 t}\right](x).$$

By our **convolution/multiplication** property, it is necessary that

$$u(t,x) = \left(\mathcal{F}^{-1}[\hat{u}_0]\right) * \left(\mathcal{F}^{-1}\left[e^{-\alpha^2 \xi^2 t}\right]\right)(x).$$

But of course,

$$\mathcal{F}^{-1}[\hat{u}_0](x) = u_0(x).$$

Second,

$$\mathcal{F}^{-1}\left[e^{-\alpha^2 \xi^2 t}\right](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\alpha^2 \xi^2 t} e^{i\xi x}\, d\xi.$$

Change of variables: letting $\xi \to -\alpha\sqrt{t}\,\xi = s$. So, $ds = -\alpha\sqrt{t}\, d\xi$. So,

$$\begin{aligned}
\mathcal{F}^{-1}\left[e^{-\alpha^2 \xi^2 t}\right](x) &= \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha\sqrt{t}} \int_{\mathbb{R}} e^{-s^2} e^{-ix(s/\alpha\sqrt{t})}\, ds, \qquad \text{we've added (-) \& flipped bounds} \\
&= \frac{1}{\alpha\sqrt{t}} \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-s^2} e^{i(x/\alpha\sqrt{t})s}\, ds\right] \\
&= \frac{1}{\alpha\sqrt{t}} \left[\mathcal{F}[e^{-s^2}]\left(\frac{x}{\alpha\sqrt{t}}\right)\right] \\
&= \frac{1}{\alpha\sqrt{t}} \frac{1}{\sqrt{2}} e^{-(x/\alpha\sqrt{t})^2/2^2} \\
&= \frac{1}{\alpha\sqrt{2t}} e^{-x^2/4\alpha^2 t} \\
&= H_t(x).
\end{aligned}$$

This is called the **heat kernel**.

So,

$$\begin{aligned} u(t,x) &= (u_0 * H_t)(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_0(y) H_t(x-y)\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_0(y) \frac{1}{\alpha\sqrt{2t}} e^{-(x-y)^2/4\alpha^2 t}\, dy \\ &= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} u_0(y) e^{-(x-y)^2/4\alpha^2 t}\, dy. \end{aligned}$$

For small t, and $\alpha = 1$

$$u(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} u_0(y) e^{-(x-y)^2/4t}\, dy,$$

which concentrates the integral around $x = y$, so that

$$u(t,x) \approx \int u_0(y)\delta(x-y)\, dy = u_0(x).$$

Precisely,

$$\lim_{t\to 0} u(t,x) = u_0(x).$$

For large time, $u(t,x) \approx 0$. Precisely, heat diffuses to zero:

$$\lim_{t\to\infty} u(t,x) = 0.$$

Just a recap of what we've done so far: We have studied the heat equation on the full real line $\mathbb{R}$:

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \qquad ,t>0, x\in\mathbb{R} \\ u(0,x) &= u_0. \end{aligned}$$

We've found with the help of the FT,

$$u(t,x) = (H_t * u_0)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H_t(x-y) u_0(y)\, dy$$

where

$$H_t(x) = \frac{1}{\sqrt{2t}} e^{-x^2/4\alpha^2 t}$$

is called the **heat kernel**, which has many important prperties useful in analysis, geometry, mathematical physics, etc. Recall that we used to have an infinite series, now we have an integral - which we can think of as a limit of this infinite series.

Chapter 13

Laplace Transform and Application

Now we move on to the Laplace transform and then a nice PDE which is well-adapted to it. Recall, for a function $f : [0, \infty) \to \mathbb{R}$,

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt,$$

which is easy to compute. This map is invertible on a certain class of functions (vector space). So, in particular, it is one-to-one, and so $\mathcal{L}[f] = \mathcal{L}[g] \implies f = g$. This is a basis for using tables of Laplace transforms.

It also have an explicit formula for the inverse

$$\mathcal{L}^{-1}[F(s)](t) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} e^{st} F(s)\, ds,$$

which is in fact a complex line integral, which is often somewhat difficult to compute.

One upside to $\mathcal{L}$ is that it can handle functions f which do not have $\lim_{t\to\infty} f(t) = 0$, which is a requirement for the FT. The class of functions $\mathcal{L}$ can handle is called **exponentially bounded**. More precisely, if given an $f : [0, \infty) \to \mathbb{R}$, s.t

$$|f(t)| \leq Me^{at}$$

for any $t, M > 0, a \in \mathbb{R}$, then $\mathcal{L}[f](s)$ makes sense for all $s > a$.

Example 13.0.1. Let $f(t) = 1$. Then

$$\mathcal{L}[1](s) = \int_0^t e^{-st}\, dt = -\frac{-e^{st}}{s}\Bigg|_{t=0}^{t=\infty} = \frac{1}{s},$$

defined for $s > 0$.

Example 13.0.2. Let $f(t) = t$. Then

$$\mathcal{L}[f](s) = \int_0^\infty te^{-st}\,dt = -\frac{te^{-st}}{s}\bigg|_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\,dt = \frac{1}{s}\frac{1}{s} = \frac{1}{s^2}.$$

Using integration by parts, it is easy to see that

$$\mathcal{L}[\sin(\omega t)](s) = \frac{1}{\omega^2 + s^2}.$$

Proposition 13.0.1. Given $u(t, x)$, set

$$U(s, x) = \mathcal{L}_t[u(t, x)](s),$$

then the following are true

1. $\mathcal{L}[u_t(t, x)](s) = sU(s, x) - u(0, x)$.

   Proof.

$$\begin{aligned}\mathcal{L}[u_t](s) &= \int_0^\infty u_t e^{-st}\,dt \\ &= u(t, x)e^{-st}\bigg|_0^\infty - \int_0^\infty u(t, x)se^{-st}\,dt \\ &= sU(s, x) - u(0, x).\end{aligned}$$

   $\square$

2. $\mathcal{L}[u_{xx}(t, x)](s) = \frac{\partial^2}{\partial x^2}U(s, x)$.
   The proof is left as an exercise.

Definition 13.0.1. For exponentially bounded functions f, g, we define

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

to be the **convolution** .

Proposition 13.0.2.

$$\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g].$$

The proof is in Marlow's.

Example 13.0.3.

$$\mathcal{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+1}\right](t) = \mathcal{L}[1]\mathcal{L}[\sin(t)] = (1*\sin)(t).$$

We can compute

$$(1*\sin)(t) = \int_0^t 1\cdot\sin(\tau)\,d\tau = -\cos(\tau)\Big|_0^t = 1-\cos(t).$$

We can use the LT to solve a heat conduction problem. Consider

$$\begin{cases} u_t = u_{xx}, \quad & x>0, t>0 \\ u_x(t,0)-u(t,0)=0 \quad & t>0 \\ u(0,x)=u_0, \quad & x\geq 0. \end{cases}$$

We will write

$$u(s,x) = \mathcal{L}_t[u(t,x)](s),$$

and solve the problem.

$$\mathcal{L}_t[u_t](s) = s\mathcal{L}[u]-u(0,x) = sU(s,x)-u_0.$$

We also know that

$$\mathcal{L}_t[u_{xx}] = \frac{\partial^2}{\partial x^2}U(s,x).$$

So we have

$$\frac{\partial^2}{\partial x^2}U(s,x) = sU(s,x)-u_0.$$

In other words,

$$\frac{\partial^2}{\partial x^2}U(s,x)-sU(s,x) = -u_0.$$

To solve, we seek a particular solution.

$$U_p(s,x) = Ax+B.$$

So,

$$\frac{\partial^2}{\partial x^2}U_p(s,x)-sU_p(s,x) = 0-s(Ax+B) = -u_0,$$

which has to hold for all $x>0$. So we conclude $A=0$ and so $B=u_0/s$. So,

$$U_p(s,x) = \frac{u_0}{s}.$$

And given that the homogeneous solution is

$$U_h(s,x) = C_1 e^{x\sqrt{s}} + C_2 e^{-x\sqrt{s}}.$$

So the general solution is

$$U(s,x) = C_1 e^{x\sqrt{s}} + C_2 e^{-x\sqrt{s}} + \frac{u_0}{s}.$$

As we physically cannot expect that $u(t,x) \to \pm\infty$ as $x \to \infty$, we set $C_1 = 0$. So the solution is

$$U(s,x) = Ce^{-x\sqrt{s}} + \frac{u_0}{s}.$$

Now we appeal to the BCs, to which we apply the Laplace transform:

$$U(s,0) = \mathcal{L}_t[u(t,0)](s) = \mathcal{L}[u_x(t,0)](s) = \frac{\partial}{\partial x} U(s,0).$$

So, in view of the general solution,

$$C + \frac{u_0}{s} = -\sqrt{s} C e^{-0\cdot\sqrt{s}},$$

so,

$$C + \sqrt{s} C = -\frac{u_0}{s},$$

or

$$C = -\frac{u_0}{s} \frac{1}{1+\sqrt{s}}.$$

All together,

$$U(s,x) = -\frac{u_0}{s} \frac{1}{1+\sqrt{s}} e^{-x\sqrt{s}} + \frac{u_0}{s} = -u_0 \left(\frac{1}{s+s^{3/2}} e^{-x\sqrt{s}} - \frac{1}{s} \right).$$

We look at the table of Laplace inverses and find

$$u(t,x) = u_0 - u_0 \left(\operatorname{erfc}\left(\frac{x}{2\sqrt{t}} \right) - \operatorname{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}} e^{x+t} \right) \right)$$

with

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, d\xi.$$

Chapter 14

Duhamel's Principle

Idea: The convolution allows you to push forward information concerning time-dependent boundary conditions.

Hard problem:

$$\begin{aligned} u_t = u_{xx}, \qquad 0 < x < 1, t > 0u(t,0) = 0 \\ u(t,1) = f(t), \qquad t > 0 \\ u(0,x) = 0, \qquad x \in [0,1]. \end{aligned}$$

Easy problem:

$$\begin{aligned} w_t = w_{xx}, \qquad 0 < x < 1, t > 0w(t,0) = 0 \\ w(t,1) = 1, \qquad t > 0 \\ w(0,x) = 0, \qquad x \in [0,1]. \end{aligned}$$

We can use the Laplace transform (half-line problem) to solve the Easy problem.

$$W(s,x) - \mathcal{L}[w(t,x)](s).$$

Plugging in gives

$$sW(s,x) - W(0,x) = \frac{\partial^2 W}{\partial x^2}(s,x).$$

By the IC:

$$\frac{\partial^2 W}{\partial x^2}(s,x) - sW(s,x) = 0.$$

Here,

$$W(s,x) = C_1 e^{x\sqrt{s}} + C_2 e^{-x\sqrt{s}}.$$

We can apply the BCs to get

$$W(s,0) = \mathcal{L}[w(t,0)](s) = \mathcal{L}[0] = 0$$

and

$$W(s,1) = \mathcal{L}[w(t,1)](s) = \mathcal{L}[1](s) = \frac{1}{s}.$$

So,

$$W(s,0) = C_1 + C_2 = 0.$$

So, $C_1 = C = -C_2$. Thus,

$$W(s,x) = Ce^{x\sqrt{s}} - Ce^{-x\sqrt{s}}.$$

And of course,

$$W(s,1) = \frac{1}{s} = C\left(e^{x\sqrt{s}} - e^{-x\sqrt{s}}\right).$$

Thus,

$$C = \frac{1}{s\left(e^{\sqrt{s}} - e^{-\sqrt{s}}\right)}.$$

All together,

$$W(s,x) = \frac{1}{s}\left(\frac{e^{x\sqrt{s}} - e^{-x\sqrt{s}}}{e^{\sqrt{s}} - e^{x\sqrt{s}}}\right) = \frac{1}{s}\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}.$$

By tables,

$$w(t,x) = \mathcal{L}^{-1}[W(s,x)](t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(n\pi)^2 t}\sin(n\pi x).$$

We mirror this with the hard problem:

$$U(s,x) = \mathcal{L}[u(t,x)](s)$$

and

$$F(s) = \mathcal{L}[f(t)](s).$$

By exactly the same approach, we obtain

$$\begin{aligned} U(s,x) &= F(s)\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})} \\ &= F(s)s\frac{1}{s}\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})} \\ &= F(s)\cdot sW(s,x) = F(s)\cdot\mathcal{L}[w_t(t,x)](s). \end{aligned}$$

So,

$$U(s,x) = F(s) \cdot \mathcal{L}[w_t(t,x)](s) = \mathcal{L}[f(t)](s) \cdot \mathcal{L}[w_t(t,x)](s).$$

So, by the convolution property,

$$u(t,x) = f * w_t = \int_0^t f(t-\tau) w_t(\tau,x)\, d\tau.$$

We can integrate by parts, which gives us

$$u(t,x) = f(0)w(x,t) + \int_0^t f'(t-\tau) w(\tau,x)\, d\tau,$$

where $w(\tau,x)$ is given exactly by the solution to the easy problem.

Chapter 15

Elliptic problems - Laplace - Poisson - Helmholtz

Fundamental object: The Laplacian is defined in Cartesian coordinates as

$$\boxed{\Delta u = \nabla^2 u = u_{xx} + u_{yy}}$$

In polar coordinates: $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}(y/x)$

$$\boxed{\Delta u = \nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}}$$

Why are the two description equivalent? Just the chain rule! If

$$u(x, y) = U(r, \theta) = U(r(x, y), \theta(x, y)),$$

then we have

$$\begin{aligned}
u_{xx} = \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2}{\partial x^2} U(r(x, y), \theta(x, y)) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} \\
&= \frac{\partial^2 U}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial^2 U}{\partial r \partial \theta}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x^2} \\
&\quad + \frac{\partial^2 U}{\partial r^2}\left(\frac{\partial \theta}{\partial x}\right)^2 \frac{\partial^2 U}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} \\
&= U_{rr}\left(\frac{\partial r}{\partial x}\right)^2 + 2U_{r\theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + U_r \frac{\partial^2 r}{\partial x^2} + U_{\theta\theta}\left(\frac{\partial \theta}{\partial x}\right)^2 + U_\theta \frac{\partial^2 \theta}{\partial x^2}.
\end{aligned}$$

Now, add this to u_{yy}. So,

$$\nabla^2 u = U_{rr}\left(\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right) + 2U_{r\theta}\left(\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}\right)$$
$$+U_r\left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right) + U_{\theta\theta}\left(\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right).$$

Calculating the partials, we should get

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$$
$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$
$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{3/2}}$$
$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{(x^2+y^2)^{3/2}}$$
$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}$$
$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$$
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{\sqrt{(x^2+y^2)^2}}$$
$$\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{\sqrt{(x^2+y^2)^2}}$$

so that

$$\nabla^2 u = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}$$

as claimed.

Often, our problem will be posed on a circle/disk, and so solving $\nabla^2 u = 0$ is easiest in polar coordinates.

Summary, in 2D

$$\boxed{\nabla^2 u = u_{xx} + u_{yy} = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}}$$

In 3D, in Cylindrical coordinates

$$\boxed{\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + U_{zz}}$$

In Spherical coordinates:

$$\boxed{\nabla^2 u = U_{rr} + \frac{2}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + \frac{\cot\theta}{r^2}U_\theta + \frac{1}{r^2\sin^2\theta}U_{\theta\theta}}$$

Why is the Laplacian important? By studying second-order difference quotients, we'll find that

- If at a point $(x, y) \in \mathbb{R}^2$, $\nabla^2 u(x, y) > 0$, then $u(x, y)$ is **less than** the average of u among its neighboring points.
- If at a point $(x, y) \in \mathbb{R}^2$, $\nabla^2 u(x, y) < 0$, then $u(x, y)$ is **greater than** the average of u among its neighboring points.
- If at a point $(x, y) \in \mathbb{R}^2$, $\nabla^2 u(x, y) = 0$, then $u(x, y)$ is **equal to** the average of u among its neighboring points.

What does this mean? Let's say the average of u is

$$\bar{u} = \frac{1}{\text{circumference}} \int_{circle} u(x, y)\, ds.$$

Then if $\nabla^2 u = 0$, then $u(x, y) = \bar{u}$. How does this translate physically?

1. For the heat equation:

$$u_t = \alpha^2 \nabla^2 u.$$

   Let's suppose that $\nabla^2 u > 0$ at (x, y). Then the temperature of (x, y)'s neighbors is higher, no average. So, $u_t > 0$, i.e., the temperature at (x, y) increases in time. This is what we expect. The neighbors heat (x, y) up.

2. $\nabla^2 u$ as a "relaxation measure."

We will be primarily focused on three problems:

1. **Laplace's equations**:

$$\boxed{\nabla^2 u = 0}$$

   In the view of the relaxing measure interpretation, this says we seek a function u which at all points (x, y), $u(x, y)$ is equal to the average of u among all its neighboring points, i.e., u is "lazy."

   We will pose Laplace's equation along with various boundary conditions to form boundary value problems.

   Laplace's equations arise in a number of contexts:

   - Steady-state solution to the heat equation.

- Electrostatics: If there is no **charge** in a region Ω, then the electric potential will satisfy Laplace's equation on Ω.
- In Newtonian gravity: If there is no **mass** in a region Ω, then the gravitational potential will satisfy Laplace's equation on Ω.

2. **Poisson's equations:**

$$\boxed{\nabla^2 u = f}$$

This says that u will be related to the average value among its neighbors based on f (the sign of f).

Contexts:

- Steady state to $u_t = \nabla^2 u - f(x, y)$.
- Electrostatics: $\nabla^2 u = -\rho$, which describes the electric potential in a region with charge density ρ.
- Similarly with Newtonian gravity.
- **Helmholtz equations: (Eigen-equation)**

$$\nabla^2 u + \lambda u = 0$$

This arises in understanding the "modes" of the vibration of a drum(head).

15.1 Boundary Value Problems for Laplace's Equation

A boundary value problem for Laplace's equation in a region Ω with boundary $\partial\Omega$ asks for a function u satisfying

1. $\nabla^2 u = 0$ inside Ω, or for all $(x, y) \in \Omega$
2. some knowledge of u along $\partial\Omega$. This is called a boundary condition.

Example 15.1.1. Steady-state solutions to IBVPs for the heat equation. Consider

$$\begin{cases} u_t = u_{xx} + \sin(\pi x), & t > 0, 0 < x < 1 \\ u(t, 0) = u(t, 1) = 0, & \forall t > 0 \\ u(0, x) = \sin(3\pi x), & x \in [0, 1] \end{cases}$$

Provided it exists, the steady state solution is a solution for which $u_t = 0$. This is in general a function $u = u(x)$. Assuming that $u_t = 0$, we have that

$$u_{xx} = -\sin(\pi x)$$
$$u(0) = u(1) = 0.$$

With the interpretation that $u_t = 0$ is gotten by $t \to \infty$, the IC shouldn't matter. Trivially, we see that

$$u(x) = \frac{1}{\pi^2}\sin(\pi x).$$

So this works. A note: **a steady state solutions doesn't always exist**. It is possible that u_t doesn't go to 0 as $t \to \infty$. □

We study 3 principal types of boundary conditions (BC) for Laplace's equation. These are

1. **Dirichlet Conditions**

   - Simplest of all: specifies value of $u(x, y)$ along $\partial\Omega$

     Example 15.1.2. Ω = unit disk. $\partial\Omega$ = unit circle:

     $$\begin{cases} \nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_\theta\theta = 0, & \Omega = \{0 < r < 1, \theta \in [0, 2\pi]\} \\ u(1, \theta) = f(\theta), & \forall\theta \in [0, 2\pi] \end{cases}$$

     This is also called the **interior Dirichlet problem** □

     Example 15.1.3. Ω = outside of umit disk. $\delta\Omega$ = unit circle.

     $$\begin{cases} \nabla^2 u = 0, & r > 1, \theta \in [0, 2\pi] \\ u(1, \theta) = f(\theta), & \theta \in [0, 2\pi] \end{cases}$$

     This is also called the **exterior Dirichlet problem** □

     Example 15.1.4. Annulus:

     $$\begin{cases} \nabla^2 u = 0, & 1 < r < 2, \theta \in [0, 2\pi] \\ u(1, \theta) = f_1(\theta), & \theta \in [0, 2\pi] \\ u(2, \theta) = f_2(\theta), & \theta \in [0, 2\pi] \end{cases}$$

     This is also called the **mixed Dirichlet problem** □

   Remark 15.1.1.

   - Note that we ask $f(\theta)$ be periodic.
   - We will be able to solve this analytically.
   - If Ω is a bounded (it's more general, this in particular) region and $\partial\Omega$ is sufficiently nice, then the problem is well-posed, i.e. so long as $g(x, y)$ is continuous on the boundary of Ω, the problem has a unique solution. Furthermore, u depends in a reasonably nice way on g, i.e. if we modify g by a small quantity, then u only changes by a small amount.
   - Though the problem is well-posed, it's only for special regions in which a solution can be written down explicitly.

- In cases where the region is not so nice, numerical methods are often very helpful.

  Example 15.1.5. Consider:

$$\begin{cases} \nabla^2 u = 0 \text{ on } \Omega \\ u(\text{outer square}) = u_1 \\ u(\text{inner square}) = u_2 \\ \Omega = \text{square annulus} \end{cases}$$

2. **Neumann Conditions**

   Let Ω be a "region" with smooth boundary $\partial\Omega$ which has unit normal vector $\vec{n}$. Given a function $g : \partial\Omega \to \mathbb{R}$, the Neumann problem asks for a function u which satisfies

$$\begin{cases} \nabla^2 u = 0 \text{ on } \Omega \\ \boldsymbol{\nabla} u \cdot \vec{n} = g \text{ on } \partial\Omega \end{cases}$$

   In contrast to the Dirichlet problem, the Neumann problem as stated is not well-posed. To make it well-posed, we must account for two things:

   - If u satisfies the problem, consider

$$v = u + c$$

     where c is a constant. Then $\nabla^2 v = 0$. Also, $\boldsymbol{\nabla} v = \boldsymbol{\nabla} u$. And thus, $\boldsymbol{\nabla} v \cdot \vec{n} = g$ on $\partial\Omega$. Thus v is a solution. So, as stated, solutions to the Neumann problem are not unique. To take care of this, we often ask for solutions to the Neumann problem satisfying

$$\int_\Omega u \, d\vec{x} = 0.$$

   - *Compatibility condition:* We must have that

$$\int_{\partial\Omega} g(\omega) \, d\omega = 0,$$

     to avoid the existence issue. Essentially, the integral of g over the boundary is 0.

     Why are these conditions necessary?

     (a) *Physical reasons:* Suppose that the Neumann problem is obtained as the steady-state problem associated with the IBVP:

$$\begin{cases} u_t = \alpha^2 \nabla^2 u & \text{on } (0, \infty) \times \Omega \\ \boldsymbol{\nabla} u \cdot \vec{n} = g, & t > 0, x \in \Omega \\ u(0, x) = u_0(x) \end{cases}$$

If $g(x)$ is non-zero, then we know that th energy across $\partial\Omega$ is either leaking or entering. If a steady-state solution is to make sense, we need to have, at all times, the net flux of heat energy through the boundary to be zero:

$$0 = \int_{\partial\Omega} \nabla u \cdot \vec{n} = \int_{\partial\Omega} g.$$

(b) *Mathematical*: Recall the **divergence theorem:** If f, g are twice differentiable, in Ω and Ω has smooth boundary $\partial\Omega$ then

$$\int_{\Omega} (\nabla^2 f) g \, dx = \int_{\partial\Omega} g(x) \nabla f \cdot \vec{n} \, ds - \int_{\Omega} \nabla g \cdot \nabla f \, dx.$$

Provided that u solves the IBVP, by setting $f = u$ and $g = 1$, the divergence theorem says

$$\int_{\Omega} \nabla^2 u \, dx = \int_{\partial\Omega} \nabla u \cdot \vec{n} \, ds - \int_{\Omega} \nabla u \cdot \nabla 1 \, dx = \int_{\partial\Omega} \nabla u \cdot \vec{n} \, ds.$$

So, by the IBVP,

$$0 = \int_{\partial\Omega} \nabla u \cdot \vec{n} \, ds = \int_{\partial\Omega} g \, ds.$$

Example 15.1.6. Consider

$$\begin{aligned} \nabla^2 u &= 0 \qquad \text{on } \Omega = \text{unit circle} \\ \nabla u \cdot \vec{n} &= \frac{\partial u}{\partial r}(1, \theta) = \sin(2\theta), \qquad \text{for all} \theta \in [0, 2\pi]. \end{aligned}$$

Question: does g meet the compatibility condition? YES!

$$\int_{\partial\Omega} g \, ds = \int_0^{2\pi} \sin(2\theta) \, d\theta = 0$$

There's a solution. In fact, we can verify that it is

$$U(r, \theta) = \frac{-r^2 \cos(2\theta)}{2}.$$

Indeed,

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \cos(2\theta) - \cos(2\theta) + 2\cos(2\theta) = 0.$$

3. **Robin Conditions (Mixed Type)**

All are described in terms of the knowledge of u along $\partial\Omega$.

Chapter 16

Dirichlet Problem on the Circle

$$\begin{cases} \nabla^2 = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1, \theta \in [0, 2\pi] \\ u(1, \theta) = g(\theta), \quad \theta \in [0, 2\pi] \end{cases}$$

To solve, we separate variables.

$$u(r, \theta) = R(r)\Theta(\theta).$$

Plugging this in, we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

and so

$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = k.$$

We obtain

$$\begin{aligned} &r^2R'' + rR' - kR = 0 \\ &\Theta'' + k\Theta = 0. \end{aligned}$$

The ansatz is

$$R(r) = r^n.$$

So,

$$n(n-1)r^2r^{n-2} + nrr^{n-1} - kr^n = 0,$$

i.e.,

$$(n(n-1)+n-k)r^n = 0 \iff n^2 - k = 0.$$

Since we seek periodic solution, we want

$$k = m^2$$

so that

$$\Theta(\theta) = A\cos(m\theta) + B\sin(m\theta)$$

where $m \in \mathbb{N}$. So, we have determined $k = m^2$ where $m = 0, 1, 2, \ldots$. So $n = \pm m$. So, our solution, for each $m = 0, 1, 2, \ldots$ is

$$R(r) = Ar^m + Br^{-m}.$$

Here we rule out the r^{-m} solution for $m = 1, 2, \ldots$ (since we have singularity at $r = 0$). Hence, for each $m = 0, 1, 2, \ldots$ we have the solution

$$\boxed{u_n(r,\theta) = r^m(A_m \cos m\theta + B_m \sin m\theta)}$$

So we expect the general solution to be of the form

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)}$$

And we also want

$$g(\theta) = u(1,\theta).$$

So this gives us

$$g(\theta) = u(1,\theta) = \sum_{n=0}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$$

This equation will hold provided that A_n's and B_n's are the Fourier sine and cosine coefficients for g, i.e.,

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\,d\alpha \\
A_n &= \frac{1}{\pi}\int_0^{2\pi} \cos(n\alpha)g(\alpha)\,d\alpha \\
B_n &= \frac{1}{\pi}\int_0^{2\pi} \sin(n\alpha)g(\alpha)\,d\alpha.
\end{aligned}$$

So given the problem

$$\begin{cases} \nabla^2 = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < 1, \theta \in [0, 2\pi] \\ u(1,\theta) = g(\theta), & \theta \in [0, 2\pi] \end{cases}$$

the equation

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)}$$

solves the IBVP provided the A_n, B_n's are chosen as above.

Note: if Ω is a disk of radius R, then the solution is instead

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n (A_n \cos n\theta + B_n \sin n\theta)}$$

where the constants A_n, B_n's are once again given by the exact same formulas.

We would like to write this solution another way that directly involves g

$$\begin{aligned} u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\,d\alpha + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \bigg(\frac{1}{\pi}\int_0^{2\pi} &\cos(n\alpha)g(\alpha)\cos n\theta\,d\alpha \\ &+\frac{1}{\pi}\int_0^{2\pi}\sin(n\alpha)g(\alpha)\sin n\theta\,d\alpha\bigg). \end{aligned}$$

Now we can just write this a single integral:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\left(g(\alpha) + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n g(\alpha)\left(\cos(n\alpha)\cos n\theta + \sin(n\alpha)\sin n\theta\right)\right)\,d\alpha.$$

And so,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \left(\cos(n\alpha)\cos n\theta + \sin(n\alpha)\sin n\theta\right)\right)\,d\alpha.$$

Now, let

$$\begin{aligned}
P &= 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n(\cos(n\alpha)\cos n\theta + \sin(n\alpha)\sin n\theta) \\
&= 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos(n(\theta-\alpha)) \\
&= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left(e^{in(\theta-\alpha)+e^{-in(\theta-\alpha)}}\right) \\
&= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n e^{in(\theta-\alpha)} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n e^{-in(\theta-\alpha)} \\
&= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}e^{i(\theta-\alpha)}\right)^n + \sum_{n=1}^{\infty}\left(\frac{r}{R}e^{-i(\theta-\alpha)}\right)^n.
\end{aligned}$$

We notice that this is a geometric series and it converges on Ω where $r/R < 1$. So,

$$P = 1 + \frac{\frac{r}{R}e^{i(\theta-\alpha)}}{1-\frac{r}{R}e^{i(\theta-\alpha)}} + \frac{\frac{r}{R}e^{-i(\theta-\alpha)}}{1-\frac{r}{R}e^{-i(\theta-\alpha)}}$$

which we can simplify to

$$\begin{aligned}
P &= 1 + \frac{re^{i(\theta-\alpha)}}{R-re^{i(\theta-\alpha)}} + \frac{re^{-i(\theta-\alpha)}}{R-re^{-i(\theta-\alpha)}} \\
&= \frac{R^2 - 2R\cos(\theta-\alpha) + r^2 + rRe^{i(\theta-\alpha)} - r^2 + rRe^{-i(\theta-\alpha)} - r^2}{D} \\
&= \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\alpha) + r^2}.
\end{aligned}$$

So,

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)P\,d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\alpha)+r^2}g(\alpha)\,d\alpha \\
&= \boxed{\frac{1}{2\pi}\int_0^{2\pi}P(\theta-\alpha)g(\alpha)\,d\alpha}
\end{aligned}$$

where

$$P(\theta) = \frac{R^2-r^2}{R^2-2rR\cos\theta+r^2}.$$

This is called the **convolution on torus**.

Observe that at $r = 0$, $P(\theta) = 1$,

$$u(0, \phi) = \int_0^{2\pi} g(\alpha)\, d\alpha = \text{average of } g.$$

Conclusion: $u(r, \theta)$ is always equal to the average value along any circle centering at (r, θ).

$$\text{Poisson kernel} = \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \alpha) + r^2}$$

This tells us something about physical systems; namely, that the potential at a point is the *weighted average* of neighboring potentials. The Poisson kernel tells just how much *weight* to assign each point (Figure 33.2).

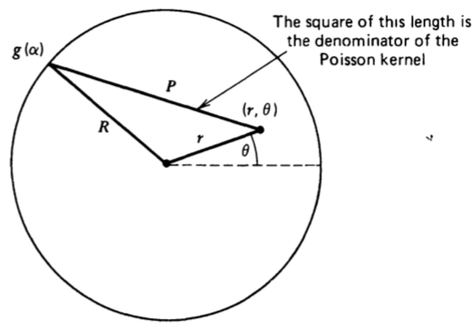


FIGURE 33.2 $u(r,\theta)$ as a weighted sum of boundary potentials.

16.1 Dirichlet's Lift

Suppose you to solve Poisson's equation

$$(1) \qquad \begin{cases} \nabla^2 u = f \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}.$$

If we can find any function V such that $\nabla^2 V = f$, then we will set

$$u = V - W,$$

so that

$$f = \nabla^2 u = \nabla^2 V - \nabla^2 W = f - \nabla^2 W.$$

Then we can solve the related problem

$$(2) \qquad \begin{cases} \nabla^2 W = 0 \text{ on } \Omega \\ W = V \text{ on } \partial\Omega \end{cases}$$

and we'd have the solution to Poisson's problem.

How about the other way? Suppose we want to solve

$$(3) \qquad \begin{cases} \nabla^2 u = 0 \text{ on } \Omega \\ u = f \text{ on } \partial\Omega \end{cases}.$$

If we can seek a function $V : \Omega \to \mathbb{R}$ such that $V = f$ along the boundary, then we can solve the related problem

$$(4) \qquad \begin{cases} \nabla^2 W = \nabla^2 V \text{ on } \Omega \\ W = 0 \text{ on } \partial\Omega \end{cases}.$$

Note $u = V - W$ solves the original problem.

16.2 Some other domains

16.2.1 Exterior of a Circle:

Consider

$$\begin{cases} \nabla^2 u = 0 \text{ on } \Omega \\ u = g \text{ on } \partial\Omega \end{cases}$$

where Ω is the exterior of a circle of radius R and $\partial\Omega$ is the circle of radius R. Once again, we separate variables.

$$u(r, \theta) = R(r)\Theta(\theta),$$

and thus

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

So,

$$\frac{r^2 R'' + rR'}{R} = -k = \frac{-\Theta''}{\Theta}.$$

So,

$$\begin{cases} r^2 R'' + rR' = -kR \\ \Theta'' - k\theta = 0, \end{cases}$$

where we have shown that $k = -n^2$, n is integer. So, for $n \neq 0$

$$\begin{cases} \Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta) \\ R(r) = C_n r^n + D_n r^{-n} \end{cases}.$$

But for $n = 0$,

$$R(r) = C_0 + D_0 \ln(r).$$

So we expect solutions

$$\begin{aligned} u(r,\theta) = &\sum_{n=0}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) b_0 \ln_n + \\ &+ \sum_{n=1}^{\infty} \alpha_n r^{-n} \cos(n\theta) + \beta_n r^{-n} \sin(n\theta). \end{aligned}$$

Now, we will reject the r^n and $\ln(r)$ solutions:

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} \alpha_n r^{-n} \cos(n\theta) + \beta_n r^{-n} \sin(n\theta)}$$

To determine the constants, we identify this as a Fourier series. Recall that

$$g(\theta) = \sum_{n=0}^{\infty} R^{-n}(\alpha_n \cos(n\theta) + \beta_n \ sin(n\theta)).$$

By Fourier's trick:

$$\begin{aligned} \alpha_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\, d\theta \\ \alpha_n &= R^n \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos(n\theta)\, d\theta \\ \beta_n &= R^n \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin(n\theta)\, d\theta. \end{aligned}$$

16.2.2 Annulus:

Now, Ω is an annulus between R_1 and R_2 where $R_1 < R_2$, and $\partial\Omega R_{1_O} = \cup R_{2_O}$. and

$$\begin{cases} \nabla^2 u = 0 \\ u(R_1,\theta) = g_1(\theta) \\ u(R_2,\theta) = g_2(\theta) \end{cases}.$$

Here, we no longer rule out any basis solutions. Thus, the general solution is

$$\begin{aligned} u(r,\theta) = &\sum_{n=0}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) + b_0 \ln(r) \\ &+ \sum_{n=1}^{\infty} \alpha_n r^{-n} \cos(n\theta) + \beta_n r^{-n} \sin(n\theta). \end{aligned}$$

Now, we have to determine $a_n, b_n, \alpha_n, \beta_n$, and we will use g_1, g_2 to do this:

$$g_1(\theta) = a_0 + b_0 \ln(R_1) + \sum_{n=1}^{\infty} (a_n R_1^n + \alpha_n R_1^{-n}) \cos(n\theta) + (b_n R_1^n + \beta_n R_1^{-n}) \sin(n\theta)$$

$$g_2(\theta) = a_0 + b_0 \ln(R_2) + \sum_{n=1}^{\infty} (a_n R_2^n + \alpha_n R_2^{-n}) \cos(n\theta) + (b_n R_2^n + \beta_n R_2^{-n}) \sin(n\theta).$$

Once again, by Fourier's trick, we have a linear system to solve for a_0, b_0:

$$\frac{1}{2\pi} \int_0^{2\pi} g_1(\theta)\, d\theta = a_0 + b_0 \ln(R_1)$$
$$\frac{1}{2\pi} \int_0^{2\pi} g_2(\theta)\, d\theta = a_0 + b_0 \ln(R_2).$$

Similarly, we get another system for a_n, α_n:

$$\frac{1}{\pi} \int_0^{2\pi} g_1 \cos(n\theta)\, d\theta = a_n R_1^n + \alpha_n R_1^{-n}$$
$$\frac{1}{\pi} \int_0^{2\pi} g_2 \cos(n\theta)\, d\theta = a_n R_2^n + \alpha_n R_2^{-n}$$

And of course:

$$\frac{1}{\pi} \int_0^{2\pi} g_1 \sin(n\theta)\, d\theta = b_n R_1^n + \beta_n R_1^{-n}$$
$$\frac{1}{\pi} \int_0^{2\pi} g_2 \sin(n\theta)\, d\theta = b_n R_2^n + \beta_n R_2^{-n}.$$

Example 16.2.1. Solve:

$$\begin{cases} \nabla^2 u = 0, & 2 \le r \le 4 \\ u(2,\theta) = 1 \\ u(4,\theta) = 2 \end{cases}.$$

We know that solutions have the form:

$$u(r,\theta) = \sum_{n=0}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) + b_0 \ln(r) +$$
$$+ \sum_{n=1}^{\infty} \alpha_n r^{-n} \cos(n\theta) + \beta_n r^{-n} \sin(n\theta).$$

Solve for the $n = 0$ cases:

$$a_0 + b_0 \ln(R_1) = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1$$
$$a_0 + b_0 \ln(R_2) = \frac{1}{2\pi} \int_0^{2\pi} 2\, d\theta = 2,$$

which says,

$$\begin{aligned}
a_0 &= 2\pi - 2\pi = 0 \\
b_0 &= \frac{2-1}{\ln(4)-\ln(2)} = \frac{1}{\ln(2)}.
\end{aligned}$$

Then we solve for a_n, α_n with $n \neq 0$:

$$\begin{aligned}
\frac{1}{\pi}\int_0^{2\pi} g_1 \cos(n\theta)\,d\theta &= a_n R_1^n + \alpha_n R_1^{-n} \\
\frac{1}{\pi}\int_0^{2\pi} g_2 \cos(n\theta)\,d\theta &= a_n R_2^n + \alpha_n R_2^{-n}
\end{aligned}$$

which says

$$\begin{aligned}
\frac{1}{\pi}\int_0^{2\pi} \cos(n\theta)\,d\theta &= 0 = a_n + \alpha_n \\
\frac{1}{\pi}\int_0^{2\pi} 2\cos(n\theta)\,d\theta &= 0 = a_n 2^n + \alpha_n 2^{-n}
\end{aligned}$$

and

$$\begin{aligned}
a_n &= 0 \\
\alpha_n &= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
b_n &= 0 \\
\beta_n &= 0.
\end{aligned}$$

So the solution is

$$\boxed{u(r,\theta) = \frac{1}{\ln(2)}\ln(r) = \frac{\ln(r)}{\ln(2)}}$$

□

Example 16.2.2. Solve:

$$\begin{cases}
\nabla^2 u = 0, \quad 1 \leq r \leq 2 \\
u(1,\theta) = \sin(2\theta) \\
u(2,\theta) = \sin(\theta)
\end{cases}.$$

We know that solutions have the form:

$$\begin{aligned}
u(r,\theta) = &\sum_{n=0}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) b_0 \ln_n + \\
&+ \sum_{n=1}^{\infty} \alpha_n r^{-n}\cos(n\theta) + \beta_n r^{-n}\sin(n\theta).
\end{aligned}$$

Solve for the $n = 0$ cases:

$$a_0 + b_0 \ln(R_1) = a_0 = \frac{1}{2\pi}\int_0^{2\pi} \sin(2\theta)\, d\theta = 0$$
$$a_0 + b_0 \ln(R_2) = \frac{1}{2\pi}\int_0^{2\pi} \sin(\theta)\, d\theta = 0,$$

which says,

$$a_0 = 0$$
$$b_0 = 0$$

Then we solve for a_n, α_n with $n \neq 0$:

$$\frac{1}{\pi}\int_0^{2\pi} g_1 \cos(n\theta)\, d\theta = a_n R_1^n + \alpha_n R_1^{-n}$$
$$\frac{1}{\pi}\int_0^{2\pi} g_2 \cos(n\theta)\, d\theta = a_n R_2^n + \alpha_n R_2^{-n}$$

which says

$$\frac{1}{\pi}\int_0^{2\pi} \sin(2\theta)\cos(n\theta)\, d\theta = a_n + \alpha_n$$
$$\frac{1}{\pi}\int_0^{2\pi} \sin(\theta)\cos(n\theta)\, d\theta = a_n 2^n + \alpha_n 2^{-n}$$

and

$$a_n = 0$$
$$\alpha_n = 0.$$

Similarly,

$$\frac{1}{\pi}\int_0^{2\pi} \sin(2\theta)\sin(n\theta)\, d\theta = a_n + \alpha_n$$
$$\frac{1}{\pi}\int_0^{2\pi} \sin(\theta)\sin(n\theta)\, d\theta = a_n 2^n + \alpha_n 2^{-n}.$$

So every term is zero except $n = 1, 2$. For $n = 1$

$$a_1 + \alpha_1 = 0$$
$$2a_1 + \frac{1}{2}\alpha_1 = 1$$

so,

$$a_1 = \frac{2}{3}$$
$$\alpha_1 = -\frac{2}{3}.$$

Similarly, for $n = 2$:

$$\begin{aligned} a_2 + \alpha_2 &= 1 \\ 4a_2 + \frac{1}{4}\alpha_2 &= 0 \end{aligned}$$

so,

$$\begin{aligned} a_2 &= -\frac{1}{15} \\ \alpha_2 &= \frac{16}{15}. \end{aligned}$$

So, the solution is

$$\boxed{u(r,\theta) = \frac{2}{3} r \sin(\theta) - \frac{2}{3} r^{-1} \sin(\theta) - \frac{1}{15} r^2 \sin(2\theta) + \frac{16}{15} r^{-2} \sin(2\theta)}$$

$\square$

Chapter 17

Laplace Equations in Spherical Coordinates

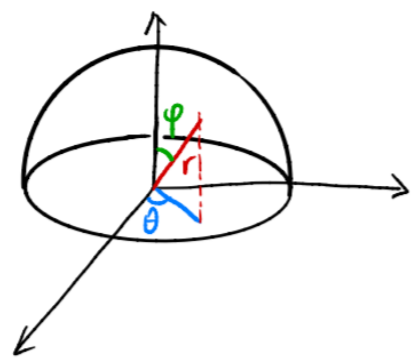


Figure 17.1: Sketch by Jerry Bao

Points in the ball of radius R are described by (r, θ, ϕ) with $0 \leq r \leq R$, $\pi \leq \theta \leq \pi$, and $0 \leq \phi \leq \pi$. In this coordinate system,

$$\begin{cases} \nabla^2 u = (r^2 u_r)_r + \frac{1}{\sin\phi}(u\phi \sin\phi)_\phi + \frac{1}{\sin^2\phi} u_{\theta\theta} = 0 \\ u(R, \theta, \phi) = g(R, \theta, \phi) \quad \text{ on } \partial\Omega = R\text{−sphere.} \end{cases}$$

Solving this problem for a general g is often difficult, so we look at special cases.

1. **Assume that** $g(\theta.\phi)$ **is constant**, i.e., spherically symmetric. We infer that $u(r, \theta, \phi) = u(r)$. Thus,

$$u_\theta = u_\phi = 0.$$

Hence the PDE now turns into an ODE:

$$\nabla^2 u = (r^2 u_r)_r = r^2 u_{rr} + 2ru_r = 0.$$

Change of variables: let $v = u_r$, then

$$\begin{aligned} r^2 v_r + 2rv &= 0 \\ v_r + \frac{2}{r} v &= 0 \\ \left(ve^{2\ln(r)}\right)' &= 0 \\ ve^{2\ln(r)} &= C \\ vr^2 &= C \\ v &= \frac{C}{r^2}. \end{aligned}$$

And so,

$$u_r = Cr^{-2}.$$

Hence

$$\boxed{u = \frac{-C}{r} + b}$$

For the problem in the ball, $C = 0$ for solution to be bounded. Thus,

$$u(r) = b.$$

Now, matching this with the boundary conditions, we get

$$u(1) = g.$$

And so the solution is

$$u(r, \theta, \phi) = g = \text{constant}.$$

So, for example, if we were to charge a metal sphere with 10 volts: To find the interior electric field, we solve Laplace's equation

$$\begin{cases} \nabla^2 u = 0. \\ u(r) = 10\text{V} \end{cases},$$

which gives

$$\vec{E} = \nabla u(r, \theta, \phi) = 0$$

everywhere inside the ball.

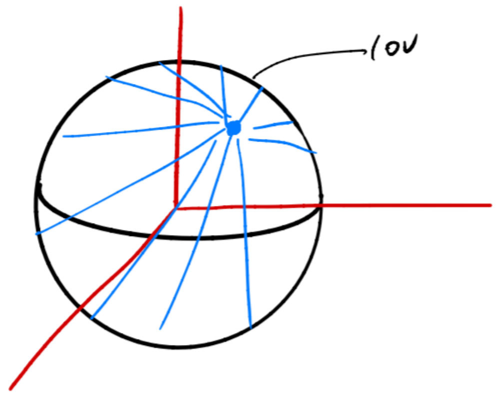


Figure 17.2: Sketch by Jerry Bao

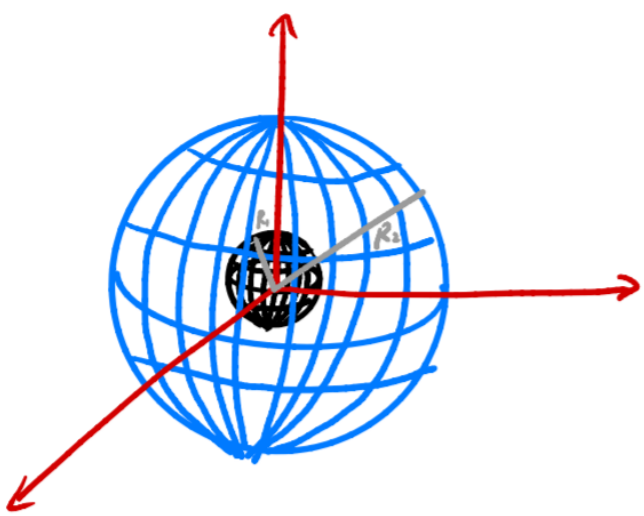


Figure 17.3: Spherical shells by Jerry Bao

2. **Spherical shells:**

   Suppose we have the problem

$$\begin{cases} \nabla^2 u = 0 \\ u(R_1, \theta, \phi) = A \\ u(R_2, \theta, \phi) = B \end{cases}$$

   where Ω is the region between the shells of radii R_1 and R_2 respectively.

We know that solutions have the form

$$u(r) = \frac{a}{r} + b.$$

And so,

$$\begin{cases} \frac{a}{R_1} + b = A \implies a + bR_1 = AR_1 \\ \frac{a}{R_2} + b = B \implies a + bR_2 = BR_2 \end{cases}.$$

Subtracting the top by the bottom gives

$$b = \frac{BR_2 - AR_1}{R_2 - R_1}.$$

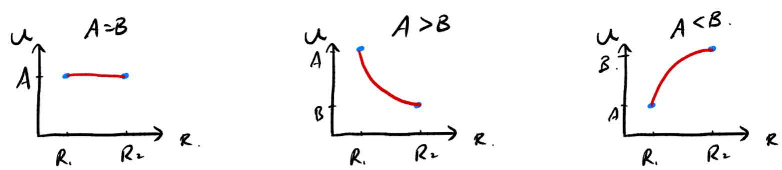
So,

$$a = AR_1 - \frac{BR_1R_2 - AR_1^2}{R_2 - R_1} = \frac{(A-B)R_1R_2}{R_2 - R_1}.$$

Therefore,

$$\boxed{u(r) = \frac{(A-B)R_1R_2}{r(R_2 - R_1)} + \frac{BR_2 - AR_1}{R_2 - R_1}}$$

We have the following sketches (credit to Jerry Bao):



3. **Assume that** $\Omega = R-$**ball and** $g = g(\phi)$ **where** $0 \le \phi \le \pi$. Here, $u(r, \theta, \phi) = u(r, \phi)$ and we have

$$\begin{cases} (r^2 u_r)_r + \frac{1}{\sin\phi}(u_\phi \sin\phi)_\phi = 0 \\ u(R, \phi) = g(\phi) \end{cases}.$$

We once again solve using separation of variables. Assume that

$$u(r, \theta) = R(r)\Phi(\phi).$$

Subjecting this to the PDE:

$$(r^2R')'\Phi + \frac{1}{\sin\phi}(\Phi' \sin\phi)' R = 0$$

or equivalently,

$$\frac{(r^2R')'}{R} = -\frac{(\Phi' \sin\phi)'}{\Phi \sin\phi} = n(n-1)$$

where $n(n+1) = k$. This gives two ODE's:

$$\boxed{r^2R'' + 2rR' + n(n+1)R = 0}$$

which is called the **Euler's equations** and

$$\boxed{(\Phi' \sin\phi)' + n(n+1)\Phi \sin\phi = 0}$$

which is called the **Legendre's equation**. Solutions to the Euler's equation is:

$$R(r) = ar^n + br^{-(n+1)}.$$

To solve the Legendre's equation, we do a substitution: $x = \cos\phi$ to get:

$$\boxed{(1-x^2)\frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx} + n(n+1)\Phi = 0}$$

where $-1 \leq x \leq 1$. To obtain the (one) bounded solution, it has to be the case that $n \in \mathbb{N}$. The solutions are

$$\begin{aligned}
&P_0(x) = 1 \\
&P_1(x) = x \\
&P_2(x) = \frac{1}{2}(3x^1 - 1) \\
&\ldots \\
&\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right)}
\end{aligned}$$

These are called the Legendre's Polynomials, and the last formula is called Rodriguez's Formula.

So we have that

$$\Phi(\phi) = a_n P_n(x) = a_n P_n(\cos\phi)$$

are solutions, and so by the principle of superposition we expect

$$\boxed{u(r,\phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi)}$$

where we have reject solutions with r^n by boundedness. What about BC? We want that:

$$g(\phi) = u(1, \phi) = \sum_{n=0}^{\infty} a_n P_n(\cos\phi).$$

So,

$$\int_0^\pi g(\phi) P_m(\cos\phi) \sin\phi \, d\phi = \sum_{n=0}^{\infty} \int_0^\pi a_n P_n(\cos\phi) P_m(\cos\phi) \sin\phi \, d\phi.$$

With Rodriguez's Formula:

$$\begin{aligned} \int_0^\pi P_n(\cos\phi) P_m(\cos\phi) \sin\phi \, d\phi &= \int_{-1}^{1} P_n(x) P_m(x) \, dx \\ &= \begin{cases} \frac{2}{2m+1} & n = m \\ 0 & \text{else} \end{cases}. \end{aligned}$$

So, for $\Omega = 1-$ball, and $\partial\Omega = 1-$sphere, the solution to

$$\left\{ \nabla^2 u = 0 \text{ on } \Omega u(1, \theta, \phi) = g(\phi) \right.$$

is given by

$$\boxed{u(r, \phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi)}$$

where

$$\boxed{a_n = \frac{2n+1}{2} \int_0^\pi g(\phi) P_n(\cos\phi) \sin\phi \, d\phi}$$

and

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x^2 - 1)^n \right)}$$

Chapter 18

Poisson's Problem

Suppose we have the problem

$$\begin{cases} \nabla^2 u = f(r,\theta) & \Omega = \text{ unit circle}, \theta \in [0,2\pi], 0 \le r \le 1 \\ u(1,\theta) = 0 & \forall \theta \in [0,2\pi] \end{cases} .$$

Let us look at a couple of interpretations of this. The solution is the steady-state temperature with time-independent heat source f and zero temperature on the boundary. Another interpretation is electrostatics. Essentially the solution is the electric potential in Ω given a charged source of $-f(r,\theta)$, while the boundary is grounded.

What is the general idea? We consider an easier problem with point sources. Suppose that we're looking at a circle of radius R with a single point source of charge $-q$. We want to understand what happens along the boundary of the circle. The total electrical energy inside the circle is q, and this should be equal to the total flux over the boundary:

$$q = -\int_0^{2\pi} u_r(r) r \, d\theta = -2\pi u_r(r) \cdot r.$$

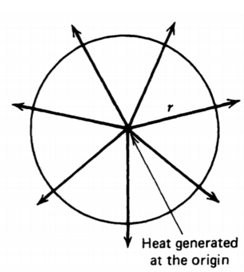


FIGURE 36.1 Radial flow of heat due to a point source.

So, the solution is

$$u_r(r) = \frac{-q}{2\pi r}.$$

And we find that the potential due to this point charge is

$$u(r) = \frac{-q}{2\pi}\ln(r) = \frac{q}{2\pi}\ln\left(\frac{1}{r}\right).$$

So, thinking in terms of charge density, we can forget about q and focus on the function $(1/2\pi)\ln(1/r)$. Going to geometry at hand. Suppose that a point charge $q = 1$ is placed inside the circle at (ρ, ϕ), while we're standing at point (r, θ). The distance R between us and the charge is

$$R = \sqrt{r^2 - 2r\rho\cos(\theta - \phi) + \rho^2}$$

by the law of cosine. We know that

$$u_{test}(r, \theta) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right)$$

satisfies the Poisson's equation with point charge at ρ, ϕ, but it won't satisfy the zero boundary condition. So, how do we get the boundary conditions to be satisfied?

To get zero boundary conditions, we imagine another charge. Here is the idea: equipotential lines. Placing the charge on the outside creates constant potential lines which are circles.

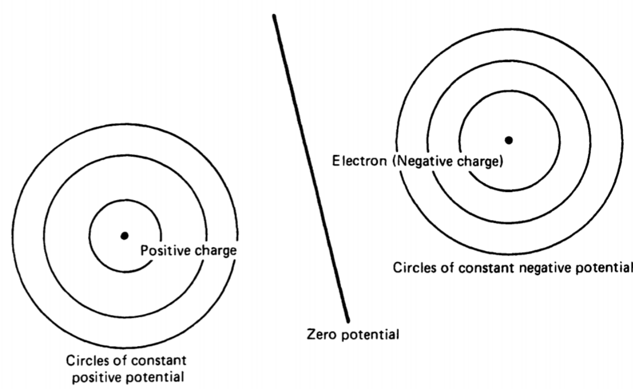


FIGURE 36.4 Potential field due to two oppositely charged particles.

Method of images: Place a charge $q = -1$ at $(1/\rho, \phi)$ making the circle of radius 1 a line of constant potential.

$$R = \sqrt{r^2 - 2r\rho\cos(\theta - \phi) + \rho^2}$$
$$\bar{R} = \sqrt{r^2 - \frac{2r}{\rho}\cos(\theta - \phi) + \frac{1}{\rho^2}}.$$

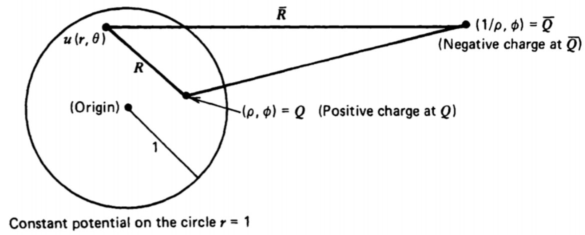


FIGURE 36.5 Charges at Q and $\bar{Q}$ giving rise to constant potential at $r = 1$.

Our new "candidate" solution becomes:

$$\boxed{u_c(r,\theta) = \frac{1}{2\pi}\ln\left(\frac{1}{R}\right) - \frac{1}{2\pi}\ln\left(\frac{1}{\bar{R}}\right)}$$

The claim is that $u_c(1,\theta)$ is constant for all θ. Let us investigate u_c on the boundary:

$$\begin{aligned}
u_c(1,\theta) &= \frac{1}{2\pi}\ln\left(\frac{\bar{R}}{R}\right) \\
&= \frac{1}{2\pi}\cdot\frac{1}{2}\cdot\ln\left(\frac{1-\frac{2}{\rho}\cos(\theta-\phi)+\frac{1}{\rho^2}}{1-2\rho\cos(\theta-\phi)+\rho^2}\right) \\
&= \frac{1}{4\pi}\ln\left(\frac{1}{\rho^2}\cdot\frac{1-\frac{2}{\rho}\cos(\theta-\phi)+\frac{1}{\rho^2}}{\frac{1}{\rho^2}-\frac{2}{\rho}\cos(\theta-\phi)+1}\right) \\
&= \frac{1}{4\pi}\ln\left(\frac{1}{\rho^2}\right) \\
&= \frac{1}{2\pi}\ln\left(\frac{1}{\rho}\right)
\end{aligned}$$

which is constant and independent of θ. So then our "adapted candidate"

$$G(r,\theta,\rho,\phi) = u_{adapted} = \frac{1}{2\pi}\ln\left(\frac{1}{R}\right) - \frac{1}{2\pi}\ln(\bar{R}) - \frac{1}{2\pi}\ln\left(\frac{1}{\rho}\right)$$

which we can also write as

$$\begin{aligned}
G(r,\theta,\rho,\phi) &= \frac{1}{2\pi}\ln\left(\frac{\rho\bar{R}}{R}\right) \\
&= \frac{1}{2\pi}\ln\left(\frac{\rho\sqrt{r^2-2\frac{r}{\rho}\cos(\theta-\rho)+\frac{1}{\rho^2}}}{\sqrt{r^2-2r\rho\cos(\theta-\rho)+\rho^2}}\right) \\
&= \boxed{\frac{1}{4\pi}\ln\left(\frac{\rho^2r^2-2r\rho\cos(\theta-\phi)+1}{r^2-2r\rho\cos(\theta-\rho)+\rho^2}\right)}
\end{aligned}$$

So, given a point charge of charge 1 at ρ, ϕ, $G(r,\theta,\rho,\phi)$ satisfies

$$\begin{cases} \nabla^2 u = \text{ Delta function } = \begin{cases} 1 & \text{at } (\rho,\phi) \\ 0 & \text{else} \end{cases} \\ u(1,\theta) = 0 \end{cases}$$

The next step is building up to $f(r,\theta)$ by assigning this "bumping" function to the value f at every point. This means

$$\boxed{u(r,\theta) = \int_0^1 \int_{-\pi}^{\pi} G(r,\theta,\rho,\phi) f(\rho,\phi) \rho \, d\phi \, d\rho}$$

We see that $u(1,\theta) = 0$ and $\nabla^2 u = 0$ on Ω. This gives the solution, and we call G the **Green's function**.

Chapter 19

Summary

1. The heat equation (parabolic problems)

$$\frac{\partial u}{\partial t} = \nabla^2 u.$$

and

$$\begin{cases} u_t = u_{xx} \\ u(0,x) = u_0(x) \end{cases} .$$

When $\Omega = \mathbb{R}$ we found that

$$\boxed{u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} u_0(y) e^{\frac{-(x-y)^2}{4t}}\, dy}$$

2. Laplace and Poisson (elliptic problems)

$$\begin{cases} \nabla^2 u = 0 \\ u = g \end{cases}$$

which are also time-independent.

3. The wave equation (hyperbolic problem)

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \nabla^2 u \qquad x \in \mathbb{R} \\ u(t,x) &= u_0(x) \qquad t > 0. \end{aligned}$$

Observe that if

$$u(t,x) = u_0(x-t),$$

called the **d'Lambert's solution**, and assuming that $u_0(x) \in \mathbb{C}^2$,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-u_0'(x-t)\right) = u_0''(x-t),$$

and

$$\nabla^2 u = \frac{\partial^2}{\partial t^2} u_0(x-t) = u_0''(x-t).$$

So this is a solution.

Here we're assuming that u_0 is twice differentiable. u moves at finite speed (finite speed of propagation), and remains localized. There's also no smoothing - in contrast to the heat equation.

Consider the heat solution

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} u_0(y) e^{\frac{-(x-y)^2}{4t}} \, dy$$

is a bell curve under the BC of the wave equation (say $u_0(x)$ is a square pulse). However, this has infinite speed of propagation and does not remain localized. The heat equation has smoothing property.

Chapter 20

Problems and Solutions

20.1 Problem set 1

Exercise.

Problem. 2, Lesson 2. The heat equation is

$$u_t = \alpha^2 u_{xx} + 1, \text{ with } 0 < x < 1.$$

Suppose $u(0,t) = 0$ and $u(1,t) = 1$. What is the steady-state temperature of the rod?

Solution. Stead-state temperature can be found by setting $u_t(x,t) = 0$ for $0 < x < 1$. It follows that $\alpha^2 u_{xx}(x,t) + 1 = 0$. In addition, the temperature profile is no longer time-dependent, so $u(x,t) \to u(x)$. These conditions give

$$\begin{aligned} u_{xx}(x) &= -\frac{1}{\alpha^2} \\ u(x) &= -\frac{1}{2\alpha^2}x^2 + Cx + D. \end{aligned}$$

Applying the boundary conditions $u(0,t) = 0$ and $u(1,t) = 1$, we can find C and D:

$$\begin{cases} u(0) = 0 = D \\ u(1) = -\frac{1}{2\alpha^2} + C = 1 \end{cases}.$$

So, $C = 1 + 1/s\alpha^2$. The temperature profile of the rod is then

$$u_{\text{steady-state}}(x) = -\frac{1}{2\alpha^2}x^2 + \left(1 + \frac{1}{2\alpha^2}\right)x.$$

Problem. 3, Lesson 2. The heat equation is

$$u_t = \alpha^2 u_{xx} - \beta u, \text{ with } 0 < x < 1.$$

Suppose the BC is $u(0,t) = 1$ and $u(1,t) = 1$. What is the steady-state temperature of the rod?

Solution. Again, we set $u_t = 0$ to find the steady-state temperature profile. This forces $\alpha^2 u_{xx} - \beta u = 0$, i.e., $\alpha^2 u_{xx} = \beta u$. Next, since the temperature is no longer time-dependent, we can let $u(x,t) \to u(x)$. Now, because β and α^2 are both positive numbers, the solution to this ODE has the form

$$u_{\text{steady-state}} = u(x) = Ce^{-\sqrt{\frac{\beta}{\alpha^2}}x} + De^{\sqrt{\frac{\beta}{\alpha^2}}x}.$$

Let us denote $\sqrt{\beta/\alpha^2}$ as ϕ. To find the coefficients C and D, we apply the boundary condition:

$$\begin{cases} u(0) = C + D = 1 \\ u(1) = Ce^{-\phi} + De^{\phi} = 1. \end{cases}$$

Solving this linear system of equation in Mathematica we find

$$C = \frac{e^\phi}{1 + e^\phi}$$
$$D = \frac{1}{1 + e^\phi}.$$

So, the steady-state temperature profile is

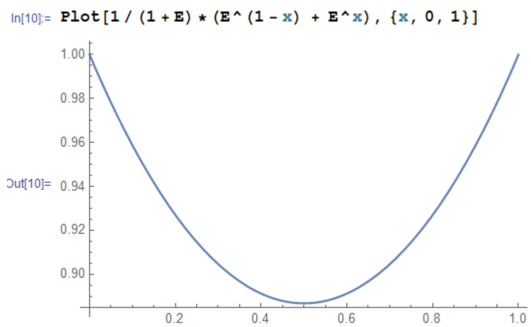
$$u_s(x) = \frac{e^\phi}{1+e^\phi}e^{-\phi x} + \frac{1}{1+e^\phi}e^{\phi x} = \frac{1}{1+e^\phi}\left(e^{\phi(1-x)} + e^{\phi x}\right),$$

where

$$\phi = \sqrt{\frac{\beta}{\alpha^2}}.$$

Mathematica code and graph of steady-state temperature distribution:

```
In[3]:= Solve[C * E^(-p) + (1 - C) * E^p == 1, C]

Out[3]= {{C -> e^p/(1 + e^p)}}

In[5]:= Simplify[1 - E^p / (1 + E^p)]

Out[5]= 1/(1 + e^p)
```



Exercise.

Problem. 1, Lesson 3. Sketch the solution to the IBVP (Farlow, 3.6) for different values of time. Check if they agree with the boundary conditions. What is the steady-state temperature of the rod? Is this obvious?

The IBVP:

$$\begin{cases} PDE : u_t = \alpha^2 u_{xx}, x \in (0, 200), t \in (0, \infty) \\ BC_1 : u_x(0, t) = 0, t \in (0, \infty) \\ BC_2 : u_x(200, t) = -\frac{h}{k}[u(200, t) - 20], t \in (0, \infty) \\ IC : u(x, 0) = 0, x \in [0, 200] \end{cases}$$

Solution 20.1.1. Sketches:

Intuitively, the steady-state temperature of the rod is just 20°C, since the rod in the problem, which is initially at 0°C, is simply being warmed up by the 20°C water. We can of course show this mathematically. By the steady-state condition, $u_t = 0 = u_{xx}$. This forces $u_{xx} = Cx + D$. But by the first boundary condition $u_x(0, t) = 0$, require that $C = 0$. The second boundary condition requires that $u_{s,x} = C = 0 = (-h/k)[u(200, t) - 20] = (-h/k)[200C + D - 20]$, which means $D = 20$. So, the steady-state temperature profile, not surprisingly, is 20°C uniform along the length of the rod.

Problem. 2, Lesson 3. Interpret the IBVP:

$$\begin{cases} PDE: u_t = \alpha^2 u_{xx}, x \in (0,1), t \in (0,\infty) \\ BC_1: u(0,t) = 0, t \in (0,\infty) \\ BC_2: u_x(1,t) = 1, t \in (0,\infty) \\ IC: u(x,0) = \sin(\pi x), x \in [0,1] \end{cases}$$

Solution 20.1.2.

Interpretation: The PDE suggests that we are dealing with heat flow in one dimension, so we can imagine a rod of length 1 with no laterally heat transfer. The first boundary condition suggests that the temperature is held fixed at 0 at $x = 0$ for all t. The second boundary condition suggests that temperature is increasing (at a constant rate) at $x = 1$ end. The initial condition tells us that initially, the temperature profile of the rod has a sinusoidal distribution across the rod's length, with the ends having temperature of 0 ($\sin(0) = \sin(\pi) = 0$) and the middle $x = 1/2$ having the highest temperature of 1.

Steady-state? The steady-state condition requires that $u_{xx} = 0$, so again, we have $u_s(x) = Cx + D$, where C, D are real constants. The first boundary condition requires $D = 0$. The second boundary condition requires that $u_x(1) = C \times 1 = C = 1$. Therefore, in the long run, $u_s(x) = x$. So, the steadstate temperature at each point of the rod has the same value as the position (from the 0 degree end) of that point on the rod.

Sketches:

Problem. 3, Lesson 3. Interpret the following IBVP:

$$\begin{cases} PDE: u_t = \alpha^2 u_{xx}, x \in (0,1), t \in (0,\infty) \\ BC_1: u_x(0,t) = 0, t \in (0,\infty) \\ BC_2: u_x(1,t) = 0, t \in (0,\infty) \\ IC: u(x,0) = \sin(\pi x), x \in [0,1] \end{cases}$$

Solution 20.1.3.

Interpretation: The PDE suggests that we are dealing with heat flow in one dimension, so we can imagine a rod of length 1 with no laterally heat transfer. The boundary conditions suggest that there are no temperature gradients at the ends of the rod. So we imagine the rod being insulated at the ends. The initial condition is like that in the previous problem where the temperature profile of the rod has a sinusoidal distribution across the rod's length, with the ends being at zero degrees ($\sin(0) = \sin(\pi) = 0$) and the middle $x = 1/2$ having the highest temperature of 1.

Steady-state: The steady-state condition requires that $u_{xx} = 0$, i.e., $u_s(x) = Cx + D$, where C, D are real constants. Since the temperatures are fixed at the end points, $u_{s,x} = C = 0$. So the steady state temperature is D, which takes some value between 0 and 1 as $t \to \infty$. The steady-state temperature profile is the same along the length of the rod.

Sketches:

Exercise.

Problem. 3, Lesson 4. Derive the heat equation

$$u_t = \frac{1}{c\rho}\partial_x[k(x)u_x] + f(x,t)$$

for the situation where the thermal conductivity $k(x)$ depends on x.

Solution. We can start the derivation from step (4.2) in Farlow's, modify $k \to k(x)$. The conservation of energy gives:

$$c\rho A \int_x^{x+\Delta x} u_t(s,t)\,ds = A\left(k(x+\Delta x)u_x(x+\Delta x,t) - k(x)u_x(x,t) + \int_x^{x+\Delta x} f(s,t)\,ds\right).$$

By the MVT, there exists $\zeta \in (x, x+\Delta x)$ such that

$$c\rho u_t(\zeta,t)\Delta x = k(x+\Delta x)u_x(x+\Delta x,t) - k(x)u_x(x,t) + f(\zeta,t)\Delta x,$$

i.e.,

$$u_t(\zeta,t) = \frac{1}{c\rho}\left\{\frac{k(x+\Delta x)u_x(x+\Delta x,t) - k(x)u_x(x,t)}{\Delta x}\right\} + \frac{1}{c\rho}f(\zeta,t)$$

Letting $\Delta x \to 0$, we turn the term with Δx into a derivative of a composition defined as $UK(x,t) = k(x)u_x(x,t)$. The result is

$$u_t(x,t) = \frac{1}{c\rho}\partial_x(k(x)u_x(x,t)) + f(x,t),$$

where we simply let $f(x,t)$ absorb the constant $(c\rho)^{-1}$. We have obtained the heat equation with x-dependent thermal conductivity.

Exercise.

Problem. 1, Lesson 5. Show that

$$u(x,t) = e^{-\lambda^2\alpha^2 t}(A\sin\lambda x + B\cos\lambda x)$$

satisfies the PDE $u_t = \alpha^2 u_{xx}$ for $A, B, \lambda \in \mathbb{R}$.

Solution. We can compute the partial derivatives and verify that $u(x,t)$ solves the PDE "by inspection." The t-derivative gives the same $u(x,t)$, multiplied by a factor of $-\lambda^2\alpha^2$, while the x-second derivative also gives $u(x,t)$, but multiplied by factor of λ^2. So, these expressions differ by a factor of α^2. Mathematically:

$$u_t = -\lambda^2\alpha^2 u(x,t) = \alpha^2 u_{xx}.$$

Hence, $u(x,t)$ solves the given PDE.

Problem. 2, Lesson 5. Let δ_n^m denotes the Kronecker delta, where m, n are non-negative whole numbers. Show

$$\int_0^1 \sin(\pi m x)\sin(\pi n x)\,dx = \frac{1}{2}\delta_n^m$$

Solution. Applying the hinted trigonometric identity, we have

$$\begin{aligned}\int_0^1 \sin(\pi m x)\sin(\pi n x)\,dx &= \frac{1}{2}\int_0^1 \cos[(m-n)\pi x] - \cos[(m+n)\pi x]\,dx \\ &= \frac{1}{2}\int_0^1 \cos[(m-n)\pi x]\,dx - \frac{1}{2}\int_0^1 \cos[(m+n)\pi x]\,dx\end{aligned}$$

At this point, we can argue why the equality given by the problem is true without much computation. The argument goes as follows. If $m = n$, then the second integral vanishes because $\cos(xk\pi)$, where k is an even number and $x \in [0, 1]$, is symmetric about $x = 1/2$ and $y = 0$. If $m \neq n$, then $m - n$ and $m+n$ are either odd or even. If they are even (and positive), then both integrals on the right hand side vanish. If they are odd, then we can assume (without loss of generality) that m is odd and n is even. This makes $\sin(\pi m x)\sin(\pi n x)$ symmetric about $x = 1/2$ and $y = 0$, so the integral also vanishes over $x \in [0, 1]$.

Problem. 5, Lesson 5. What is the solution to problem 4 in Farlow, Lesson 5 (which also requires doing 3) if the initial condition is changed to

$$u(x,0) = \sin(2\pi x) + \frac{1}{3}\sin(4\pi x) + \frac{1}{5}\sin(6\pi x)$$

Solution. We should quickly do problem 3 first. If $\Phi(x) = 1$, $x \in [0,1]$. Applying the formula for the coefficients A_n:

$$A_n = 2\int_0^1 \Phi(x)\sin(n\pi x)\,dx = 2\int_0^1 \sin(n\pi x)\,dx = \frac{2}{n\pi}(1-\cos(n\pi x)) = \begin{cases} \frac{4}{n\pi}, n \text{ odd} \\ 0, n \text{ even} \end{cases}.$$

So, the Fourier expansion for $\Phi(x) = 1$ is

$$\Phi(x) = 1 = \frac{4}{\pi}\left[\sin(\pi x) + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x) + \dots\right].$$

In problem 4, the boundary and initial conditions suggests using $\Phi(x)$ from problem 3. So, given the formula for $u(x,t)$, we simply substitute in the coefficients to generate a Fourier expansion for $u(x,t)$:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t}\sin(n\pi x) \\ &= \frac{4}{\pi}\left[e^{-(\pi)^2 t}\sin(\pi x) + \frac{1}{3}e^{-(3\pi)^2 t}\sin(3\pi x) + \frac{1}{5}e^{-(5\pi)^2 t}\sin(5\pi x) + \dots\right]. \end{aligned}$$

Back to problem 5, if the initial condition is given as $u(x,0)$ above, then we might think we have to re-do and find a new Fourier expansion. But by inspecting the form of $u(x,0)$, we can see that it is just a truncated Fourier expansion. So there is no need to find the coefficients A_n since they are already given to us. So, carefully picking out the coefficients, we get the new solution

$$u(x,t) = e^{-(2\pi)^2 t}\sin(2\pi x) + \frac{1}{3}e^{-(4\pi)^2 t}\sin(4\pi x) + \frac{1}{5}e^{-(6\pi)^2 t}\sin(6\pi x).$$

20.2 Problem set 2

Exercise. Problem 2, Lesson 6
Transform

$$PDE: u_t = u_{xx}, \qquad 0 < x < 1$$
$$BCs: \begin{cases} u(0,t) = 0 \\ u(1,t) = 1 \end{cases} \quad 0 < t < \infty$$
$$IC: u(x,0) = x^2, \qquad 0 \le x \le 1$$

to zero BCs and solve the new problem. What will the solution to this problem look like for different values of time? Does the solution agree with your intuition? What is the steady-state solution? What does the transient solution look like?

Solution.
Let $u(x,t) = U(x,t) + S(x,t)$ where $U(x,t)$ is the transient solution while $S(x,t)$ is the steady-state solution to the IBVP. To find the steady-state solution $S(x,t)$, we set $u_t = 0$ and $U(x,t) = 0$. Applying the boundary conditions, we find

$$S(x,t) = S(x) = Cx + D = x.$$

So,

$$u(x,t) = x + U(x,t).$$

Since $u_t = U_t$ and $u_{xx} = U_{xx}$, we can re-write the origin IBVP as

$$PDE: U_t = U_{xx}, \qquad 0 < x < 1$$
$$BCs: \begin{cases} U(0,t) = 0 \\ U(1,t) = 0 \end{cases} \quad 0 < t < \infty$$
$$IC: U(x,0) = x^2 - x, \qquad 0 \le x \le 1$$

The solution $U(x,t)$ to this IBVP is given by

$$U(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x),$$

where

$$A_n = 2\int_0^1 (x^2 - x)\sin(n\pi x)\,dx = 2\frac{-2 + 2\cos(n\pi) + n\pi\sin(n\pi)}{n^3\pi^3}$$
$$= \begin{cases} \frac{-8}{n^3\pi^3}, & n \text{ odd} \\ 0, & n \text{ even}. \end{cases}$$

The second equality comes from integrating in Mathematica, which can also be done with integration by parts. The transient solution is then

$$U(x,t) = -\frac{8}{\pi^3}e^{-(\pi)^2 t}\sin(\pi x) - \frac{8}{27\pi^3}e^{-(3\pi)^2 t}\sin(3\pi x) - \frac{8}{125\pi^3}e^{-(5\pi)^2 t}\sin(5\pi x) + \dots$$

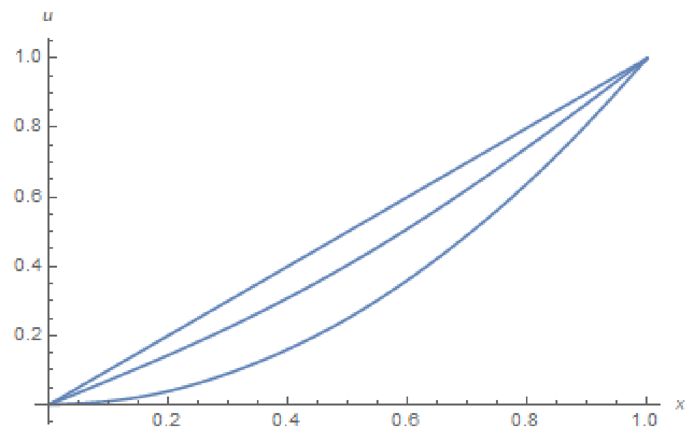
The full solution to the IBVP is

$$\begin{aligned} u(x,t) &= x - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}e^{-(n\pi)^2 t}\sin(n\pi x), \;\; n \text{ odd} \\ &= x - \frac{8}{\pi^3}e^{-(\pi)^2 t}\sin(\pi x) - \frac{8}{27\pi^3}e^{-(3\pi)^2 t}\sin(3\pi x) - \frac{8}{125\pi^3}e^{-(5\pi)^2 t}\sin(5\pi x) + \dots \end{aligned}$$

Mathematica code:

```
In[6]:= Simplify[Integrate[(x^2 - x) Sin[n*Pi*x], {x, 0, 1}]]

Out[6]= (-2 + 2 Cos[n \[Pi]] + n \[Pi] Sin[n \[Pi]])/(n^3 \[Pi]^3)

In[4]:= F[n_] := Simplify[Integrate[(x^2 - x) Sin[n*Pi*x], {x, 0, 1}]]

In[7]:= Table[F[n], {n, 1, 10}]

Out[7]= {-(4/\[Pi]^3), 0, -(4/(27 \[Pi]^3)), 0, -(4/(
125 \[Pi]^3)), 0, -(4/(343 \[Pi]^3)), 0, -(4/(729 \[Pi]^3)), 0}
```

Solutions for different values of time:



Mathematica code:

```
U[x_, t_] :=
x - (8 E^(-\[Pi]^2 t) Sin[\[Pi] x])/\[Pi]^3 - (
8 E^(-9 \[Pi]^2 t) Sin[3 \[Pi] x])/(27 \[Pi]^3) - (
8 E^(-25 \[Pi]^2 t) Sin[5 \[Pi] x])/(125 \[Pi]^3)

Show[Plot[U[x, 0], {x, 0, 1}], Plot[U[x, 0.1], {x, 0, 1}],
Plot[U[x, 1], {x, 0, 1}]]
```

The solution matches my intuition. The IBVP simply states that the temperature at the ends are fixed, and that the temperature is initially distributed across the rod as x^2. Since the temperatures at the ends are fixed it is expected that in the long run the temperature increases uniformly across the rod, in agreement with the steady-state solution.

Exercise. Problem 3, Lesson 6 Transform

$$
\begin{aligned}
&PDE : u_t = u_{xx}, \qquad 0 < x < 1 \\
&BCs : \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) + hu(1,t) = 1 \end{cases} \quad 0 < t < \infty \\
&IC : u(x,0) = \sin(\pi x), \qquad 0 \le x \le 1
\end{aligned}
$$

into a new problem with zero BCs. Is the new PDE homogeneous?

Solution.

Once again, we let $u(x,t) = U(x,t) + S(x,t)$ where $S(x,t)$ is the steady-state solution. $S(x,t)$ has the form:

$$
S(x,t) = A(t)\left(1 - \frac{x}{L}\right) + B(t)\left(\frac{x}{L}\right) = A(t)\left(1 - x\right) + B(t)\left(x\right),
$$

where

$$
\begin{aligned}
S(0,t) &= A(t) \\
S(1,t) &= B(t) \\
S_x(0,t) &= B(t) - A(t) = S_x(1,t).
\end{aligned}
$$

Applying the boundary conditions,

$$
\begin{pmatrix} -1 & 1 \\ -1 & h+1 \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

Solving the system for $A(t)$ and $B(t)$ gives

$$
A(t) = B(t) = \frac{1}{h}
$$

So, the steady-state solution is

$$
S(x,t) = \frac{1}{h}(1 - x + x) = \frac{1}{h},
$$

which is independent of t and x. Therefore, $u_t = U_t = u_{xx} = U_{xx}$. Applying the initial condition, we find

$$
u(x,0) = S(x,0) + U(x,0) = \frac{1}{h} + U(x,0) = \sin(\pi x)
$$

and

$$
u_x(1,t) + hu(1,t) = U(1,t) + hU(1,t) + \frac{h}{h} = 1.
$$

So, the new IBVP is:

$$PDE : U_t = U_{xx}, \qquad 0 < x < 1$$

$$BCs : \begin{cases} U_x(0,t) = 0 \\ U_x(1,t) + hU(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$IC : u(x,0) = \sin(\pi x) - \frac{1}{h}, \qquad 0 \leq x \leq 1$$

We notice that the new PDE is still homogeneous.

Exercise. Problem 1, Lesson 7 Solve the following heat-flaw problem:

$$PDE : u_t = u_{xx}, \qquad 0 < x < 1, 0 < t < \infty$$
$$BCs : \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} \quad 0 < t < \infty$$
$$IC : u(x,0) = x, \qquad 0 \le x \le 1$$

by separation of variables. Does your solution agree with the your intuition? What is the steady-state solution?

Solution.

By separation of variables, we assume $u(x,t) = T(t)X(x)$. By the PDE,

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \mu,$$

where μ is some constant. We reject solutions with $\mu > 0$ on physical grounds. If $\mu = 0$, then $T'(t) = X''(x) = 0$, so

$$u(x,t) = Ax + B.$$

To satisfy the boundary conditions:

$$\begin{aligned} u(0,t) &= B = 0 \\ u_x(1,t) &= A = 0. \end{aligned}$$

This $u(x,t) = 0$, a trivial solution. If $\mu < 0$, then we let $\mu = -\lambda^2$. We immediately have

$$\begin{aligned} T(t) &= Ae^{-\lambda^2 t} \\ X(x) &= C\sin(\lambda x) + B\cos(\lambda x). \end{aligned}$$

So, the general solution is

$$u(x,t) = e^{-\lambda^2 t}(A\sin(\lambda x) + B\cos(\lambda x)).$$

Subjecting $u(x,t)$ to the first boundary condition, we find $B = 0$, which reduces the solution to

$$u(x,t) = Ae^{-\lambda^2 t}\sin(\lambda x).$$

The second boundary condition gives

$$A\cos(\lambda) = 0.$$

Assuming $A \neq 0$, so that our solution is not trivial,

$$\lambda = \frac{k\pi}{2}, \qquad k \text{ odd}.$$

The solution is then

$$u(x,t) = \sum_{k=1}^{\infty} A_k e^{-(k\pi/2)^2 t} \sin\left(\frac{k\pi}{2}x\right), \quad k \text{ odd.}$$

To find the coefficients A_k, we invoke Fourier's trick, for odd n's:

$$\begin{aligned}
\int_0^1 u_0(x) \sin\left(\frac{n\pi}{2}x\right)\, dx &= \int_0^1 \sin\left(\frac{n\pi}{2}x\right) \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi}{2}x\right)\, dx \\
&= \sum_{k=1}^{\infty} A_k \int_0^1 \sin\left(\frac{n\pi}{2}x\right) \sin\left(\frac{k\pi}{2}x\right)\, dx \\
&= \sum_{k=1}^{\infty} \frac{1}{2} A_k \delta_n^k \\
&= \frac{1}{2} A_n.
\end{aligned}$$

So,

$$A_k = 2\int_0^1 x \sin\left(\frac{k\pi}{2}x\right)\, dx, \quad k \text{ odd.}$$

So, the transient solution has the form:

$$U(x,t) = \sum_{k=1}^{\infty} A_k e^{-(k\pi/2)^2 t} \sin\left(\frac{k\pi}{2}x\right), \quad k \text{ odd,}$$

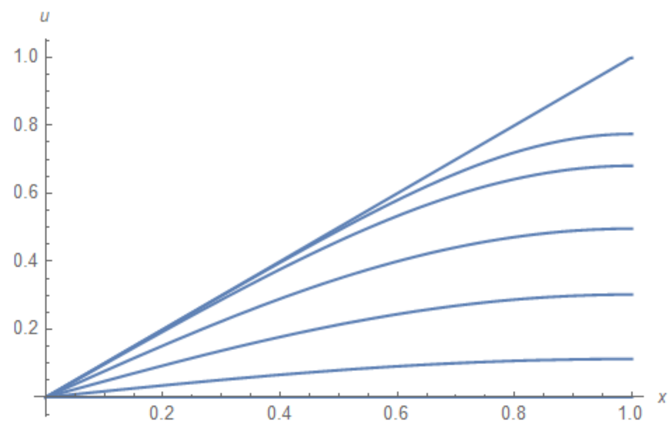
and A_k is given above. The steady-state solution is 0. So, the full solution is then

$$u(x,t) = U(x,t).$$

Mathematica code:

```
A[k_] := 2*Integrate[x*Sin[k*Pi*x/2], {x, 0.0001, 1}]

(2 (-2 k \[Pi] Cos[(k \[Pi])/2] + 4 Sin[(k \[Pi])/2]))/(k^2 \[Pi]^2)


U[x_, t_] :=
Sum[(2 (-2 (2 k - 1) \[Pi] Cos[((2*k - 1) \[Pi])/2] +
4 Sin[((2*k - 1) \[Pi])/2]))/((2 k - 1)^2 \[Pi]^2)*
E^(-t*((2 k - 1)*Pi)^2)*Sin[(2 k - 1)*Pi*x/2], {k, 1, 100}]


Show[Plot[U[x, 0], {x, 0, 1}], Plot[U[x, 0.05], {x, 0, 1}],
Plot[U[x, 0.01], {x, 0, 1}], Plot[U[x, 0.1], {x, 0, 1}],
Plot[U[x, 0.02], {x, 0, 1}], Plot[U[x, 0.2], {x, 0, 1}],
Plot[U[x, 1], {x, 0, 1}], AxesLabel -> {x, u}]
```

The figure below shows solutions for different values of time.



The solution has good sense, first because the steady-state solution is zero - as suggested by the BCs and IC. Second, at $t = 0$, $u(x, 0) = x$. Third, the boundary condition requires the temperature at $x = 0$ stay fixed and the temperature gradient at $x = 1$ to be zero, which the plots also illustrate nicely.

Exercise. Problem 2, Lesson 7 What are the eigenvalues and eigenfunctions of the Sturm-Liouville problem?

$$ODE : X'' + \lambda X = 0, \qquad 0 < x < 1$$

$$BCs : \begin{cases} X(0) = 0 \\ X'(1) = 0 \end{cases}$$

What are the functions $p(x)$, $q(x)$, and $r(x)$ in the general Sturm-Liouville problem for this equation?

Solution.

1. $p(x) = 1$.
2. $q(x) = 0$.
3. $r(x) = 1$.

To find the eigenvalues and the eigenfunctions, we have to solve the IVP. By the ODE, we know that

$$X(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right).$$

Applying the initial conditions, we find

$$B = 0$$
$$-A\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) = 0.$$

i.e., for n odd, we find the eigenvalues:

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2.$$

And so the eigenfunctions are

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{2}\right).$$

Exercise. 2. Problem 3, Lesson 7 Solve the following problem with insulated boundaries:

$$PDE: u_t = u_{xx}, \qquad 0 < x < 1, 0 < t < \infty$$
$$BCs: \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} \quad 0 < t < \infty$$
$$IC: u(x,0) = x, \qquad 0 \le x \le 1$$

Does your solution agree with your interpretation of the problem? What is the steady-state solution? Does this make sense?

Solution.

Let $u(x,t) = U(x,t) + S(x,t)$ where $S(x,t)$ is the steady-state and $U(x,t)$ is the transient solution. We can construct the steady-state solution as

$$S(x,t) = A(t)\,(1-x) + B(t)x.$$

As before, by applying the boundary conditions, we require that $A(t)$ and $B(t)$ solve the following linear system

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The system has infinitely many solutions, but we get $A(t) = B(t)$. Assuming that at steady-state, $u_t = u_{xx} = 0 = S_t$, we get

$$S(x,t) = A(t)(1-x) + A(t)x = A(t) = \Lambda$$

where Λ is constant.

Applying separation of variables to this PDE, we know that

$$T(t) = e^{-\lambda^2 t}$$
$$X_n(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

Applying the boundary conditions,

$$B = 0$$
$$\lambda_n = n\pi.$$

So, the general solution is

$$u(x,t) = \Lambda + \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t}\cos(n\pi x).$$

Next, we want to find the coefficients A_n. Since there is a cos involved, we will use Fourier's trick with a cosine and the identity

$$\int_0^1 \cos(m\pi x)\cos(n\pi x) = \frac{1}{2}\delta_n^m.$$

This gives

$$\begin{aligned}\int_0^1 u_0(x)\cos(m\pi x)\,dx &= \int_0^1 \cos(m\pi x)\sum_{n=1}^{\infty} A_n\cos(n\pi x)\,dx \\ &= \sum_{n=1}^{\infty} A_n\frac{1}{2}\delta_m^n \\ &= \frac{A_m}{2}.\end{aligned}$$

So, we can compute A_m in Mathematica (or by integration by parts):

$$\begin{aligned}A_m = 2\int_0^1 x\cos(m\pi x)\,dx &= \frac{2(\pi m\sin(\pi m)+\cos(\pi m)-1)}{\pi^2 m^2} \\ &= \frac{2(\cos(\pi m-1))}{m^2\pi^2} \\ &= \begin{cases}\frac{-4}{m^2\pi^2}, & m \text{ odd} \\ 0, & m \text{ even.}\end{cases}\end{aligned}$$

The general solution is then

$$\begin{aligned}u(x,t) &= \Lambda - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-(n\pi)^2 t}\cos(n\pi x),\ \ n \text{ odd} \\ &= \Lambda - \frac{4}{\pi^2}e^{-(\pi)^2 t}\cos(\pi x) - \frac{4}{9\pi^2}e^{-(3\pi)^2 t}\cos(3\pi x) - \frac{4}{25\pi^2}e^{-(5\pi)^2 t}\cos(5\pi x) + \dots\end{aligned}$$

To find the steady-state solution, we only need to look for Λ such that $u(0,0) = 0$ (to satisfy the initial condition), i.e.,

$$\begin{aligned}\Lambda &= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},\ \ n \text{ odd} \\ &= \frac{1}{2}.\end{aligned}$$

So steady-state solution is $S(x,t) = 1/2$, and the full solution is

$$\begin{aligned}u(x,t) &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-(n\pi)^2 t}\cos(n\pi x),\ \ n \text{ odd} \\ &= \frac{1}{2} - \frac{4}{\pi^2}e^{-(\pi)^2 t}\cos(\pi x) - \frac{4}{9\pi^2}e^{-(3\pi)^2 t}\cos(3\pi x) - \frac{4}{25\pi^2}e^{-(5\pi)^2 t}\cos(5\pi x) + \dots\end{aligned}$$

Mathematica code:

```
Simplify[Integrate[Cos[m*Pi*x]*Cos[n*Pi*x], {x, 0, 1}]]
(m Cos[n \[Pi]] Sin[m \[Pi]] - n Cos[m \[Pi]] Sin[n \[Pi]])/(
m^2 \[Pi] - n^2 \[Pi])

2*Integrate[x*Cos[m*Pi*x], {x, 0, 1}]
(2 (-1 + Cos[m \[Pi]] + m \[Pi] Sin[m \[Pi]]))/(m^2 \[Pi]^2)

A[m_] := (2 (-1 + Cos[m \[Pi]] + m \[Pi] Sin[m \[Pi]]))/(m^2 \[Pi]^2)
Table[A[m], {m, 1, 7}]
{-(4/\[Pi]^2), 0, -(4/(9 \[Pi]^2)), 0, -(4/(25 \[Pi]^2)), 0, -(4/(
49 \[Pi]^2))}

N[Sum[(4/Pi^2)*(1/(2 n - 1)^2)*1, {n, 1, 10000}]]
0.49999
```

20.3 Problem set 3

Exercise. 1. For the following equations and associated boundary conditions (together, Sturm-Liouville Problems), determine the form of the eigenfunctions and give a formula (in terms of the determinant as we did in lecture) of the associated eigenvalues λ. Find an approximate value for λ_1, the smallest eigenvalue and estimate λ_n for large values of n.

Recall from class that λ was an eigenvalue of the associated Sturm-Liouville Problem:

$$\begin{cases} \text{ODE:} & L[u](x) = \lambda r(x)u(x) \quad 0 < x < 1 \\ \text{BC1:} & a_1 u(0) + b_1 u'(0) = 0 \\ \text{BC2:} & a_2 u(1) + b_2 u'(1) = 0 \end{cases}$$

where $L[u] = -(p(x)u'(x))' + q(x)u(x)$ provided, for linearly independent solutions u_1 and u_2 to the differential equation $L[u] = \lambda r u$,

$$\det(A(\lambda)) = \det \begin{pmatrix} a_1 u_1(0) + b_1 u_1'(0) & a_1 u_2(0) + b_1 u_2'(0) \\ a_2 u_1(1) + b_2 u_1'(1) & a_2 u_2(1) + b_2 u_2'(1) \end{pmatrix} = 0.$$

---

1. $y'' - \lambda y = 0 \qquad y(0) + y'(0) = 0, y(1) = 0.$

Solution. First, we identify:

$$\begin{aligned} &p(x) = 1, \quad q(x) = 0, \quad r(x) = -1. \\ &a_1 = 1 \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = 0. \end{aligned}$$

The associated S-L problem is

$$\begin{cases} \text{ODE:} & L[y](x) = y'' = -\lambda y \quad 0 < x < 1 \\ \text{BC1:} & y(0) + y'(0) = 0 \\ \text{BC2:} & y(1) = 0 \end{cases}$$

The general solution to the ODE has the form

$$y(x) = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} e^{i\sqrt{\lambda}x} & e^{-i\sqrt{\lambda}x} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

where C_1 and C_2 are determined by the BCs, which give rise to the system

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 + i\sqrt{\lambda} & 1 - i\sqrt{\lambda} \\ e^{i\sqrt{\lambda}} & e^{-i\sqrt{\lambda}} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

To get non-trivial solutions, we require $\det(A(\lambda)) = 0$, i.e.,

$$(1 + i\sqrt{\lambda})e^{-i\sqrt{\lambda}} - (1 - i\sqrt{\lambda})e^{i\sqrt{\lambda}} = 0.$$

Let $\mathcal{L} = \sqrt{\lambda}^2$, (we're assuming λ is positive in this problem)

$$\begin{aligned} 0 &= (1 + i\mathcal{L})e^{-i\mathcal{L}} - (1 - i\mathcal{L})e^{i\mathcal{L}} \\ &= (1 + i\mathcal{L})(\cos\mathcal{L} - i\sin\mathcal{L}) - (1 - i\mathcal{L})(\cos\mathcal{L} + i\sin\mathcal{L}) \\ &= 2i(\mathcal{L}\cos\mathcal{L} - \sin\mathcal{L}) \end{aligned}$$

if and only if

$$\boxed{\tan\mathcal{L} = \mathcal{L}}.$$

Since we require $\lambda \neq 0$ is get non-trivial solutions, we only consider $\mathcal{L}_1, \mathcal{L}_2, \ldots$. Numerically approximate in Mathematica gives

$$\begin{aligned} &\mathcal{L}_1 \approx 4.493409457909114 \\ &\mathcal{L}_n \approx n. \end{aligned}$$

Converting $\mathcal{L}$ back to λ and requiring $\lambda < 0$, we get

$$\boxed{\begin{aligned} &\lambda_1 \approx 20.1907 \\ &\lambda_n \approx n^2 \qquad n \text{ large} \end{aligned}}$$

While we can't find C_1 and C_2 exactly, we can find their ratio from the linear system:

$$\frac{C_1}{C_2} = \frac{-e^{-i\sqrt{\lambda}}}{e^{i\sqrt{\lambda}}} = -e^{-2i\sqrt{\lambda}}.$$

The unnormalized eigenfunction thus has the form

$$y(x) = -e^{-2i\sqrt{\lambda_n}}e^{i\sqrt{\lambda_n}x} + e^{-i\sqrt{\lambda_n}x} = -e^{i\sqrt{\lambda_n}(x-2)} + e^{-i\sqrt{\lambda_n}x}.$$

Since $r(x) = -1$, normalizing y only involves the modulus square of it:

$$\begin{aligned} ||y||^2 &= -(-e^{i\sqrt{\lambda_n}(x-2)} + e^{-i\sqrt{\lambda_n}x})(-e^{-i\sqrt{\lambda_n}(x-2)} + e^{i\sqrt{\lambda_n}x}) \\ &= -2 + e^{i\sqrt{\lambda_n}(x-2)+i\sqrt{\lambda_n}x} + e^{-i\sqrt{\lambda_n}x-i\sqrt{\lambda_n}(x-2)} \\ &= -2 + e^{i\sqrt{\lambda_n}(x-2)} + e^{-i\sqrt{\lambda_n}(x-2)} \\ &= -2 + \cos\left(\sqrt{\lambda_n}(x-2)\right). \end{aligned}$$

The normalizing condition requires

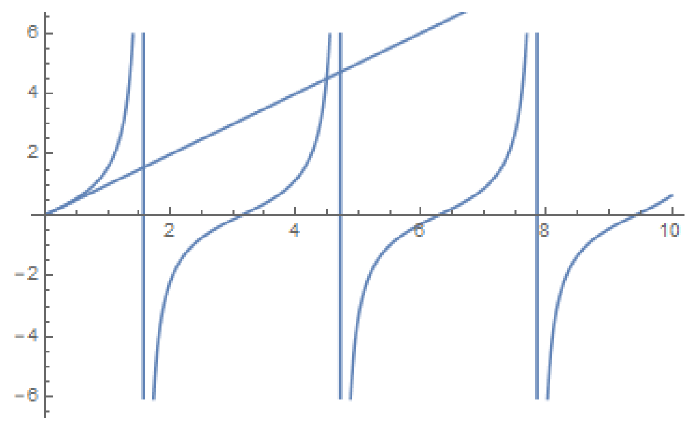
$$1 = -\int_0^1 2 - \cos\left(\sqrt{\lambda_n}(x-2)\right)\, dx.$$

Integrating in Mathematica and normalizing gives the form for an eigenfunction:

$$\boxed{y_n(x,\lambda_n) = \frac{\sqrt{\lambda_n}}{\sin\sqrt{\lambda_n}(1-2\cos\sqrt{\lambda_n})+2\sqrt{\lambda_n}}\left(e^{i\sqrt{\lambda_n}(x-2)} - e^{-i\sqrt{\lambda_n}x}\right)}$$

Mathematica code:

```
Simplify[(1 + I*L) (Cos[L] - I*Sin[L]) - (1 - I*L) (Cos[L] +
I*Sin[L])]
2 I (L Cos[L] - Sin[L])

t[L_] := Tan[L] - L; val = 0.000000000001;
FindRoot[val == t[L], {L, 4}]
{L -> 4.49341}
Show[Plot[Tan[L], {L, 0, 10}], Plot[L, {L, 0, 10}]]

Simplify[Integrate[2 - Cos[A*(x - 2)], {x, 0, 1}]]
2 + ((1 - 2 Cos[A]) Sin[A])/A
```



2. $y'' + \lambda y = 0 \quad y'(0) = 0, y(1) + y'(1) = 0.$

Solution. First, we identify:

$$\begin{aligned} &p(x) = 1, \quad q(x) = 0, \quad r(x) = 1. \\ &a_1 = 0 \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = 1. \end{aligned}$$

The associated S-L problem is

$$\begin{cases} \text{ODE:} & L[y](x) = y'' = -\lambda y \quad 0 < x < 1 \\ \text{BC1:} & y'(0) = 0 \\ \text{BC2:} & y(1) + y'(1) = 0 \end{cases}$$

The general solution to the ODE has the form

$$y(x) = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{\lambda}x & \sin\sqrt{\lambda}x \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

where C_1 and C_2 are determined by the BCs, which give rise to the system

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\lambda} \\ \cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda} & \sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

To get non-trivial solutions, we require $\det(A) = 0$, i.e.,

$$0 = -\sqrt{\lambda}(\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda}).$$

This means

$$\boxed{\tan\sqrt{\lambda} = \frac{1}{\sqrt{\lambda}}}$$

Numerically approximate in Mathematica gives

$$\boxed{\begin{aligned} &\lambda_1 \approx - - - \\ &\lambda_n \approx n \quad n \text{ large} \end{aligned}}$$

NEEDS SOME CORRECTION HERE

Exercise. 2. In lecture, we saw that the linear operator

$$L[u] = -(p(x)u'(x))' + q(x)u(x)$$

had the property that

$$\int_0^1 (vL[u] - uL[v])\,dx = 0$$

whenever u and v are twice-continuously differentiable functions which satisfy the boundary conditions of the Strum-Liouville Problem.

In class, we assumed b_1 and b_2 were non-zero. Here, show that the result remains valid provided b_1 or b_2 is zero. Do u and v have to be eigenfunctions of the S-L problem for the property above to hold? Or, is it simply required that u and v satisfied the boundary conditions? Assume the identity:

$$\int_0^1 (fL[g] - gL[f])\,dx = p(x)(g'(x)f(x) - f'(x)g(x))\Big|_0^1$$

which is true for all twice-continuously differentiable functions on the interval $[0, 1]$.

Solution. Assume that

$$\int_0^1 (fL[g] - gL[f])\,dx = p(x)(g'(x)f(x) - f'(x)g(x))\Big|_0^1,$$

and (without loss of generality) $b_1 = 0, b_2 \neq 0$, i.e., $a_1\phi(0) = 0$ and $a_2\phi(1) + b_2\phi'(1) = 0$ for $\phi = v$ or u. If this implies $\phi(1) = \phi(0) = 0$, then we have

$$\begin{aligned}
\int_0^1 (vL[u] - uL[v])\,dx &= p(x)(u'(x)v(x) - v'(x)u(x))\Big|_0^1 \\
&= p(1)(u'(1)v(1) - v'(1)u(1)) - p(0)(u'(0)v(0) - v'(0)u(0)) \\
&= p(1)(u'(1)v(1) - v'(1)u(1)) \\
&= p(1)\left(\frac{-b_2u(1)}{a_2}v(1) - \frac{-b_2v(1)}{a_2}u(1)\right) \\
&= 0.
\end{aligned}$$

If $a_1, a_2 = 0$ then there are no boundary conditions, so we reject this possibility.

Observe that the identity holds regardless of whether u and v are eigenfunctions of the S-L problem. It is simply required that u and v satisfied the boundary conditions.

Exercise. 3. Consider a second-order linear differential operator

$$M[u](x) = \kappa_2 u''(x) + \kappa_1(x)u'(x) + \kappa_0 u(x)$$

and linear homogeneous boundary conditions

$$\begin{cases} \text{BC1:} & a_1 u(0) + b_1 u'(0) = 0 \\ \text{BC2:} & a_2 u(1) + b_2 u'(1) = 0 \end{cases}$$

where $(a_1, b_1) \neq (0,0)$ and $(a_2, b_2) \neq (0,0)$; we take both the operator M and the BCs to be defined for all twice-continuously differentiable functions u on $[0,1]$. We say that the operator $M[u]$, when restricted to the BCs is *formally self-adjoint* provided

$$\int_0^1 uL[v] - vL[u]\, dx = 0$$

whenever u and v satisfy the BCs.

Are the following operators restricted to the given boundary conditions formally self-adjoint? Justify your answer.

---

1. $M[u] = u'' + u' + 2u, \qquad u(0) = u(1) = 0.$

   Solution. $M[u]$ is self-adjoint if

   $$\int_0^1 (vM[u] - uM[v])\, dx = 0.$$

   We first expand and simplify the integrand.

   $$\begin{aligned} vM[u] - uM[v] &= v(u'' + u' + 2u) - u(v'' + v' + 2v) \\ &= (vu'' - uv'') + (vu' - uv'). \end{aligned}$$

   Integrating both sides gives

   $$\int_0^1 (vM[u] - uM[v])\, dx = \int_0^1 (vu'' - uv'')\, dx + \int_0^1 (vu' - uv')\, dx.$$

   Consider the first term.

   $$\begin{aligned} \int_0^1 (vu'' - uv'')\, dx &= \int_0^1 vu''\, dx - \int_0^1 uv''\, dx \\ &= vu'\Big|_0^1 - \int_0^1 v'u'\, dx - uv'\Big|_0^1 + \int_0^1 u'v'\, dx \\ &= vu' - uv'\Big|_0^1 \\ &= (v(1)u'(1) - u(1)v'(1)) - (v(0)u'(0) - u(0)v'(0)) \\ &= 0. \end{aligned}$$

We can attempt to simplify the second term, in a similar fashion:

$$
\begin{aligned}
\int_0^1 (vu' - uv')\, dx &= \int_0^1 vu'\, dx - \int_0^1 uv'\, dx \\
&= vu\Big|_0^1 - \int_0^1 uv'\, dx - \int_0^1 uv'\, dx \\
&= v(1)u(1) - v(0)u(0) - \int_0^1 uv'\, dx - \int_0^1 uv'\, dx \\
&= -2\int_0^1 uv'\, dx \\
&\neq 0.
\end{aligned}
$$

Therefore $M[u]$ is not self-adjoint.

2. $M[u] = (1+x^2)u'' + 2xu' + u, \qquad u'(0) = u(1) + 2u'(1) = 0.$

Solution. $M[u]$ is self-adjoint if

$$\int_0^1 (vM[u] - uM[v])\,dx = 0.$$

We first expand and simplify the integrand.

$$\begin{aligned} vM[u] - uM[v] &= v((1+x^2)u'' + 2xu' + u) - u((1+x^2)v'' + 2xv' + v) \\ &= (1+x^2)(vu'' - uv'') + 2x(vu' - uv'). \end{aligned}$$

Integrating both sides gives

$$\begin{aligned} \int_0^1 (vM[u] - uM[v])\,dx &= \int_0^1 (1+x^2)(vu'' - uv'')\,dx + \int_0^1 2x(vu' - uv')\,dx. \\ &= \int_0^1 vu'' - uv''\,dx + \int_0^1 x^2(vu'' - uv'')\,dx \\ &\quad + \int_0^1 2x(vu' - uv')\,dx. \end{aligned}$$

Consider the first term:

$$\begin{aligned} \int_0^1 (vu'' - uv'')\,dx &= \int_0^1 (vu'' - uv'')\,dx \\ &= vu'\Big|_0^1 - uv'\Big|_0^1 \\ &= v(1)u'(1) - v(0)u'(0) - u(1)v'(1) + u(0)v'(0) \\ &= v(1)u'(1) - u(1)v'(1) \\ &= +2v'(1)\frac{1}{2}u(1) - u(1)v'(1) \\ &= 0. \end{aligned}$$

Consider the second term:

$$\begin{aligned} \int_0^1 x^2(vu'' - uv'')\,dx &= \int_0^1 x^2\left[(vu')' - u'v' - (uv')' + u'v'\right]\,dx \\ &= \int_0^1 x^2\left[(vu')' - (uv')'\right]\,dx \\ &= x^2(vu')\Big|_0^1 - \int_0^1 2x(vu')\,dx - x^2(uv')\Big|_0^1 + \int_0^1 2x(uv')\,dx \\ &= v(1)u'(1) - u(1)v'(1) - \int_0^1 2x(vu' - uv')\,dx \\ &= +2v'(1)\frac{1}{2}u(1) - u(1)v'(1) - \int_0^1 2x(vu' - uv')\,dx. \end{aligned}$$

Putting everything together, we find a cancellation:

$$\begin{aligned}\int_0^1 (vM[u] - uM[v])\,dx &= -\int_0^1 2x(vu' - uv')\,dx + \int_0^1 2x(vu' - uv')\,dx \\ &= 0.\end{aligned}$$

Therefore, $M[u]$ is self-adjoint.

3. $M[u] = (1+x^2)u'' + 2xu' + u, \qquad u(0) - u'(1) = u'(0) + 2u(1) = 0.$

Solution. $M[u]$ is self-adjoint if

$$\int_0^1 (vM[u] - uM[v])\,dx = 0.$$

We first expand and simplify the integrand

$$\begin{aligned} vM[u] - uM[v] &= v((1+x^2)u'' + 2xu' + u) - u((1+x^2)v'' + 2xv' + v) \\ &= (1+x^2)(vu'' - uv'') + 2x(vu' - uv'). \end{aligned}$$

Integrating both sides gives

$$\begin{aligned} \int_0^1 (vM[u] - uM[v])\,dx &= \int_0^1 (1+x^2)(vu'' - uv'')\,dx + \int_0^1 2x(vu' - uv')\,dx. \\ &= \int_0^1 vu'' - uv''\,dx + \int_0^1 x^2(vu'' - uv'')\,dx \\ &\quad + \int_0^1 2x(vu' - uv')\,dx. \end{aligned}$$

Consider the first term:

$$\begin{aligned} \int_0^1 (vu'' - uv'')\,dx &= \int_0^1 (vu'' - uv'')\,dx \\ &= vu'\Big|_0^1 - uv'\Big|_0^1 \\ &= v(1)u'(1) - v(0)u'(0) - u(1)v'(1) + u(0)v'(0). \end{aligned}$$

Consider the second term:

$$\begin{aligned} \int_0^1 x^2(vu'' - uv'')\,dx &= \int_0^1 x^2\left[(vu')' - u'v' - (uv')' + u'v'\right]\,dx \\ &= \int_0^1 x^2\left[(vu')' - (uv')'\right]\,dx \\ &= x^2(vu')\Big|_0^1 - \int_0^1 2x(vu')\,dx - x^2(uv')\Big|_0^1 + \int_0^1 2x(uv')\,dx \\ &= v(1)u'(1) - u(1)v'(1) - 2x\int_0^1 (vu' - uv')\,dx. \end{aligned}$$

Putting everything together, we find a cancellation:

$$\begin{aligned}
\int_0^1 (vM[u] - uM[v])\,dx &= -\int_0^1 2x(vu' - uv')\,dx + \int_0^1 2x(vu' - uv')\,dx \\
&\quad + 2v(1)u'(1) - v(0)u'(0) - 2u(1)v'(1) + u(0)v'(0) \\
&= 2v(1)u'(1) - v(0)u'(0) - 2u(1)v'(1) + u(0)v'(0) \\
&= -v'(0)u(0) - v(0)u'(0) + u'(0)v(0) + u(0)v'(0) \\
&= 0.
\end{aligned}$$

Therefore, $M[u]$ is self-adjoint.

Exercise. 4. Problem 3, Lesson 8
Solve

$$\text{PDE:} \quad u_t = u_{xx} - u \quad 0 < x < 1, 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{IC: } u(x,0) = \sin(\pi x) \quad 0 \le x \le 1.$$

directly by separation of variables without making any preliminary transformation. Does your solution agree with the solution you would obtain if the transformation

$$u(x,t) = e^{-t} w(x,t)$$

were made in advance?

Solution. Let

$$u(x,t) = X(x)T(t).$$

The PDE tells us that

$$T_t X = X_{xx} T - XT,$$

i.e.,

$$\frac{T_t}{T} = \frac{X_{xx}}{X} - 1 = -\lambda^2 - 1.$$

where λ is a constant. So,

$$\begin{cases} T(t) = e^{-(\lambda^2+1)t} \\ X(x) = A\cos\lambda x + B\sin\lambda x \end{cases}$$

Applying the BCs, we get

$$\begin{aligned} A &= 0 \\ \lambda &= n\pi, \qquad n = 0, 1, 2, 3, \ldots \end{aligned}$$

So the general solution (to the homogeneous case)is

$$u(x,t) = \sum_{n=0}^{\infty} e^{-((n\pi)^2+1)t} B_n \sin(n\pi x),$$

where B_n can be solved by applying the IC:

$$\begin{aligned} \int_0^1 \sin(\pi x)\sin(m\pi x)\,dx &= \int_0^1 \sum_{n=0}^{\infty} B_n \sin(n\pi x)\sin(m\pi x)\,dx \\ &= \frac{1}{2} B_n \delta_m^n \\ &= \frac{1}{2} B_m. \end{aligned}$$

This gives

$$B_n = 2\int_0^1 \sin(\pi x)\sin(n\pi x)\,dx = \delta_n^1.$$

So the solution to the homogeneous PDE is

$$\boxed{u(x,t) = e^{-(\pi^2+1)t}\sin(\pi x)}.$$

Now, if we apply the transformation $u(x,t) = e^{-t}w(x,t)$ in advance, then the extra factor of e^{-t} will be taken care of in advance, and the IBVP will be transformed into one with both homogeneous PDE and BCs that involves $w(x,t)$ rather than $v(x,t)$. First, we can look at how the PDE changes:

$$\begin{aligned} u_t &\to -e^{-t}w(x,t) + e^{-t}w_t(x,t) \\ u_{xx} - u &\to e^{-t}w_{xx}(x,t) - e^{-t}w(x,t). \end{aligned}$$

For the original PDE to hold, we require that

$$w_t = w_{xx}.$$

Since the BCs and ICs are time-independent, the transformation simply allows us to replace v by w. So, the new IBVP is

$$\begin{cases} \text{PDE:} \quad w_t = w_{xx} \quad 0 < x < 1, 0 < t < \infty \\ \text{BCs:} \quad \begin{cases} w(0,t) = 0 \\ w(1,t) = 0 \end{cases} \quad 0 < t < \infty \\ \text{IC: } w(x,0) = \sin(\pi x) \quad 0 \le x \le 1. \end{cases}$$

This is exactly how we treated $u(x,t)$ before correcting it to make it solve the PDE. So, our solution agrees with the solution we would obtain if the transformation $u(x,t) = e^{-t}w(x,t)$ were made in advance.

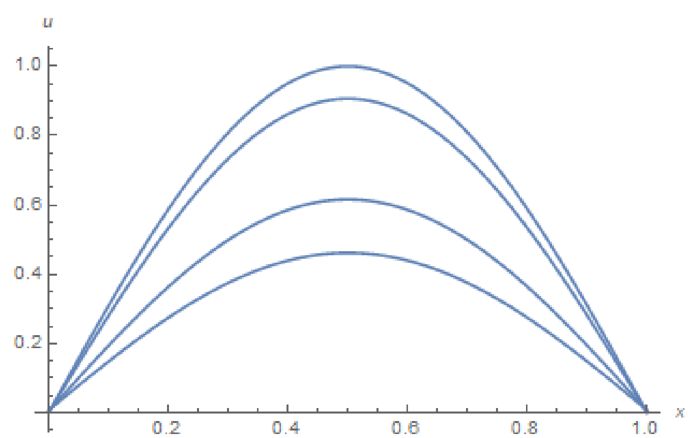
20.4 Problem set 4

Exercise. 1. Problem 1, Lesson 9, Farlow

$$u(x,t) = e^{-(\pi\alpha)^2 t}\sin(\pi x) + \frac{1}{(3\pi\alpha)^2}\left[1 - e^{-(3\pi\alpha)^2 t}\right]\sin(3\pi x).$$

Solution. We can plot $u(x,t)$ for a few time points:

```
U[t_, x_] :=
E^(-Pi^2 t)*Sin[Pi*x] +
1/(3*Pi)^2*(1 - E^(-(3 Pi)^2 t))*Sin[3 Pi*t];

Show[Plot[U[0, x], {x, 0, 1}], Plot[U[0.05, x], {x, 0, 1}],
Plot[U[0.01, x], {x, 0, 1}], Plot[U[0.08, x], {x, 0, 1}],
AxesLabel -> {x, u}]
```



The solution makes sense, because the IBVP basically describes a rod with fixed zero temperature at the ends and an initial temperature distribution of $\sin(\pi x)$ being subjected to some heat source.

Exercise. 2. Problem 2, Lesson 9, Farlow
Solve the IBVP:

$$\begin{aligned}
&\text{PDE: } u_t = u_{xx} + \sin(\pi x) + \sin(2\pi x), \quad 0 < x < 1 \ \ 0 < t < \infty \\
&\text{BCs: } \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty \\
&\text{IC: } u(x,0) = 0 \ \ 0 \le x \le 1.
\end{aligned}$$

Solution. We use the eigenfunction expansion method to solve this problem. The first step is finding the eigenvectors by converting the homogeneous problem:

$$\begin{aligned}
&\text{PDE: } u_t = u_{xx}, \quad 0 < x < 1 \ \ 0 < t < \infty \\
&\text{BCs: } \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t\infty
\end{aligned}$$

into the associated S-L problem

$$\begin{aligned}
&\text{PDE: } X_{xx} = -\lambda^2 X, \quad 0 < x < 1 \\
&\text{BCs: } \begin{cases} X(0) = 0 \\ X(1) = 0 \end{cases} \quad ,
\end{aligned}$$

to which the solutions (or the eigenvectors) have the form:

$$X_n(x) = \sin(n\pi x).$$

Next, we want to write $f(x,t) = \sin(\pi x) + \sin(2\pi x)$ in terms of the eigenvectors. Observe that we don't need Fourier Sine series here since $f(x,t)$ is already given to us as a sum of two eigenvectors:

$$f(t,x) = \sin(\pi x) + \sin(2\pi x) = \sum_{n=1}^{\infty} f_n(t) X_n(x) = f_1(t) \times X_1(x) + f_2(t) \times X_2(x).$$

This means

$$\begin{aligned}
f_1(t) &= f_2(t) = 1 \\
f_n(t) &= 0, \quad n > 2.
\end{aligned}$$

Subject

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

to the IBVP, where

$$f(t,x) = \sum_{n=1}^{\infty} f_n(t) X_n(x) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x)$$

gives

$$\dot{T}_n(t) + (n\pi)^2 T_n(t) = f_n(t),$$

to which the solution is of the form

$$T_n(t) = e^{-(n\pi)^2 t} \int_0^t f_n(s) e^{(n\pi)^2 s}\, ds + T_n(0) e^{-(n\pi)^2 t}.$$

Since we only have non-zero $f_1(t) = f_2(t) = 1$,

$$\begin{aligned} T_1(t) &= e^{-(\pi)^2 t} \int_0^t e^{(\pi)^2 s}\, ds + T_1(0) e^{-(\pi)^2 t} \\ &= e^{-(\pi)^2 t} \left(\frac{-1 + e^{\pi^2 t}}{\pi^2} + T_1(0) \right), \end{aligned}$$

and

$$\begin{aligned} T_2(t) &= e^{-(2\pi)^2 t} \int_0^t e^{(2\pi)^2 s}\, ds + T_2(0) e^{-(2\pi)^2 t} \\ &= e^{-(2\pi)^2 t} \left(\frac{-1 + e^{4\pi^2 t}}{4\pi^2} + T_2(0) \right), \end{aligned}$$

and

$$T_n(t) = T_n(0) e^{-(n\pi)^2 t}, \quad n > 2.$$

Next, note that because

$$u(0, x) = u_0(x) = 0 = \sum_{n=1}^{\infty} T_n(0) \sin(n\pi x),$$

we have

$$T_n(0) = 0,$$

which means all $T_n(t) = 0$ for $n > 2$ are zero. Therefore,

$$\begin{aligned} u(t, x) &= \sum_{n=1}^{\infty} \left(e^{-(n\pi)^2 t} \int_0^t f_n(s) e^{(n\pi)^2 s}\, ds \right) \sin(n\pi x) \\ &= e^{-(\pi)^2 t} \left(\frac{-1 + e^{\pi^2 t}}{\pi^2} \right) \sin(\pi x) + e^{-(2\pi)^2 t} \left(\frac{-1 + e^{4\pi^2 t}}{4\pi^2} \right) \sin(2\pi x) \\ &= \frac{1}{\pi^2} \left(1 - e^{-\pi^2 t} \right) \sin(\pi x) + \frac{1}{(2\pi)^2} e^{-(2\pi)^2 t} \left(1 - e^{-(2\pi)^2 t} \right) \sin(2\pi x). \end{aligned}$$

Exercise. 4. Problem 5, Lesson 9, Farlow

Solve the IBVP:

$$\text{PDE: } u_t = u_{xx} + \sin(\pi x), \quad 0 < x < 1 \;\; 0 < t < \infty$$
$$\text{BCs: } \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t < \infty$$
$$\text{IC: } u(x,0) = 1 \;\; 0 \le x \le 1.$$

Solution. We use the eigenfunction expansion method to solve this problem. The first step is finding the eigenvectors by converting the homogeneous problem:

$$\text{PDE: } u_t = u_{xx}, \quad 0 < x < 1 \;\; 0 < t < \infty$$
$$\text{BCs: } \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \quad 0 < t\infty$$

into the associated S-L problem

$$\text{PDE: } X_{xx} = -\lambda^2 X, \quad 0 < x < 1$$
$$\text{BCs: } \begin{cases} X(0) = 0 \\ X(1) = 0 \end{cases} \quad ,$$

to which the solutions (or the eigenvectors) have the form:

$$X_n(x) = \sin(n\pi x).$$

Next, we want to write $f(x,t) = \sin(\pi x)$ in terms of the eigenvectors. Observe that we don't need Fourier Sine series here since $f(x,t)$ is already given to us as a sum of two eigenvectors:

$$f(t,x) = \sin(\pi x) = \sum_{n=1}^{\infty} f_n(t) X_n(x) = f_1(t) \times X_1(x).$$

This means

$$f_n(t) = \delta_n^1.$$

Subject

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

to the IBVP, where

$$f(t,x) = \sum_{n=1}^{\infty} f_n(t) X_n(x) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x)$$

gives

$$\dot{T}_n(t) + (n\pi)^2 T_n(t) = f_n(t),$$

to which the solution is of the form

$$T_n(t) = e^{-(n\pi)^2 t} \int_0^t f_n(s) e^{(n\pi)^2 s}\, ds + T_n(0) e^{-(n\pi)^2 t}.$$

Since we only have non-zero $f_1(t) = 1$,

$$\begin{aligned} T_1(t) &= e^{-(\pi)^2 t} \int_0^t e^{(\pi)^2 s}\, ds + T_1(0) e^{-(\pi)^2 t} \\ &= e^{-(\pi)^2 t} \left(\frac{-1 + e^{\pi^2 t}}{\pi^2} + T_1(0) \right), \end{aligned}$$

and

$$T_n(t) = T_n(0) e^{-(n\pi)^2 t}, \;\; n > 2.$$

Next, note that because

$$u(0,x) = u_0(x) = 1 = \sum_{n=1}^{\infty} T_n(0) \sin(n\pi x),$$

we have

$$T_n(0) = 2 \int_0^1 \sin(n\pi x)\, dx = \begin{cases} 4/(n\pi), & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Therefore,

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \left(e^{-(n\pi)^2 t} \int_0^t f_n(s) e^{(n\pi)^2 s}\, ds \right) \sin(n\pi x) \\ &= e^{-(\pi)^2 t} \left(\frac{-1 + e^{\pi^2 t}}{\pi^2} + \frac{4}{\pi} \right) \sin(\pi x) + \sum_{n=2}^{\infty} \frac{4}{n\pi} e^{-(n\pi)^2 t} \sin(n\pi x), \;\; n \text{ odd} \\ &= \left(\frac{1 - e^{\pi^2 t}}{\pi^2} + \frac{4}{\pi} e^{-\pi^2 t} \right) \sin(\pi x) + \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} e^{-(2j+1)^2 \pi^2 t} \sin\left[(2j+1)\pi x\right] \end{aligned}$$

Exercise. 3. Problem 5, Lesson 9, Farlow

Solve:

$$
\begin{aligned}
&\text{PDE: } u_t = u_{xx} \qquad 0 < x < 1 \\
&\text{BCs: } \begin{cases} u(0,t) = 0 \\ u(1,t) = \cos t \end{cases} \qquad 0 < t < \infty \\
&\text{IC: } u(x,0) = x \qquad 0 \le x \le 1
\end{aligned}
$$

by

1. Transforming into one with zero BCs.

2. Solving the resulting problem by expanding it in terms of eigenfunctions.

Solution.

1. Transforming the IBVP into one with zero BCs.

   To do this, we consider the steady-state where $u_t = 0$, which gives $u_x x = 0$, i.e., $S(x,t) = Cx + D$. Applying the BCs, we get

   $$
   \begin{aligned}
   D &= 0 \\
   C &= \cos t.
   \end{aligned}
   $$

   So, the steady-state solution is

   $$S(x,t) = x\cos t,$$

   and the full solution is

   $$u(x,t) = U(x,t) + x\cos t.$$

   Applying the BCs to this u to obtain the BCs for U:

   $$
   \begin{aligned}
   U(0,t) &= u(0,t) - S(0,t) = 0 - 0 = 0 \\
   U(1,t) &= u(1,t) - S(1,t) = \cos t - \cos t = 0.
   \end{aligned}
   $$

   And the initial condition becomes

   $$U(x,0) = u(x,0) - S(x,0) = x - x = 0.$$

   We have transformed the BCs into one with zeros. But the PDE has turned inhomogeneous:

   $$U_t = u_t - S_t = u_{xx} + x\sin t = U_{xx} + x\sin t.$$

The new IBVP is then

$$\text{PDE: } U_t = U_{xx} + x\sin t \qquad 0 < x < 1$$
$$\text{BCs: } \begin{cases} U(0,t) = 0 \\ U(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$
$$\text{IC: } U(x,0) = 0 \qquad 0 \le x \le 1$$

2. Now we consider eigenfunction expansion. The eigenfunction to the ODE. To do this, we consider the homogeneous problem, and convert it into an associated S-L problem to get the ODE:

$$\text{PDE: } X_{xx} = -\lambda^2 X \qquad 0 < x < 1$$
$$\text{BCs: } \begin{cases} X(0) = 0 \\ X(1) = 0 \end{cases}$$

to which the eigenfunctions have the form

$$X_n(x) = \sin(n\pi x).$$

Now we want to write $x\sin t$ in terms of these eigenfunctions:

$$x\sin t = \sum_{n=1}^{\infty} f_n(t) X_n(x) = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x).$$

So integrating in Mathematica gives

$$\begin{aligned} f_n(t) &= 2\int_0^1 x\sin t\sin(n\pi x)\,dx \\ &= \begin{cases} (-1)^{n+1}\frac{2\sin t}{n\pi}, & n \ge 1 \\ 0, \quad n = 0. \end{cases} \end{aligned}$$

Now, we also have

$$\begin{aligned} U_t(x,t) &= \sum_{n=1}^{\infty} \dot{T}_n(t)\sin(n\pi x) \\ U_{xx}(x,t) &= \sum_{n=1}^{\infty} -(n\pi)^2 T_n(t)\sin(n\pi x). \end{aligned}$$

Since $U_t = U_{xx} + \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)$, we have the following ODE:

$$\dot{T}_n(t) + (n\pi)^2 T_n(t) = (-1)^{n+1}\frac{2\sin t}{n\pi},$$

whose solution is

$$\begin{aligned} T_n(t) &= e^{-(n\pi)^2 t} \int_0^\tau (-1)^{n+1} \frac{2\sin\tau}{n\pi} e^{(n\pi)^2\tau}\, d\tau + T_n(0) e^{-(n\pi)^2 t} \\ &= (-1)^{n+1} \frac{2}{n\pi} \int_0^\tau e^{-(n\pi)^2(t-\tau)} \sin\tau\, d\tau. \end{aligned}$$

Therefore, the full solution, $u(x,t) = S(x,t) + U(x,t)$, is

$$\boxed{u(x,t) = x\cos t + \sum_{n=1}^{\infty} \sin(n\pi x) \left((-1)^{n+1} \frac{2}{n\pi} \int_0^\tau e^{-(n\pi)^2(t-\tau)} \sin\tau\, d\tau \right)}$$

Mathematica code:

```
F[n_] := Simplify[Integrate[x*Sin[n*Pi*x], {x, 0, 1}]]

In[12]:= Table[F[n], {n, 0, 10}]

Out[12]= {0, 1/\[Pi], -(1/(2 \[Pi])), 1/(3 \[Pi]), -(1/(4 \[Pi])), 1/(
         5 \[Pi]), -(1/(6 \[Pi])), 1/(7 \[Pi]), -(1/(8 \[Pi])), 1/(
         9 \[Pi]), -(1/(10 \[Pi]))}
```

Exercise. 5. Problem 1, Lesson 10, Farlow Prove the identities

1. $\mathcal{F}_s[f'] = -\omega\mathcal{F}_c[f]$

2. $\mathcal{F}_s[f''] = \frac{2}{\pi}\omega f(0) - \omega^2\mathcal{F}_s[f]$

3. $\mathcal{F}_c[f'] = \frac{-2}{\pi}f(0) + \omega\mathcal{F}_s[f]$

4. $\mathcal{F}_c[f''] = \frac{-2}{\pi}f'(0) - \omega^2\mathcal{F}_c[f]$.

What assumptions do you need to make about the function f?

Solution. We have to assume that f has to be at least twice piecewise differentiable, and that $f \to 0$ as $t \to \infty$.

1. The first identity can be shown by integration by parts

$$\begin{aligned}
\mathcal{F}_s[f'] &= \frac{2}{\pi}\int_0^\infty f' \sin(\omega t)\,dt \\
&= \frac{2}{\pi}\left(f\sin(\omega t)\Big|_0^\infty - \int_0^1 \omega f\cos(\omega t)\,dt \right) \\
&= \frac{2}{\pi} f\sin(\omega t)\Big|_0^\infty - \omega\mathcal{F}_c[f] \\
&= -\omega\mathcal{F}_c[f].
\end{aligned}$$

2. We use the first identity

$$\begin{aligned}
\mathcal{F}_s[f''] &= -\omega\mathcal{F}_c[f'] \\
&= -\omega\frac{2}{\pi}\int_0^\infty f'\cos(\omega t)\,dt \\
&= -\frac{2\omega}{\pi}\left(\lim_{t\to\infty} f\cos(\omega t)\Big|_0^t + \omega\int_0^\infty f''\sin(\omega t)\,dt \right) \\
&= \frac{2}{\pi}\omega f(0) - \frac{2\omega^2}{\pi}\int_0^\infty f''\sin(\omega t)\,dt \\
&= \frac{2}{\pi}\omega f(0) - \omega^2\mathcal{F}_s[f].
\end{aligned}$$

3. We use the first and second identities:

$$\begin{aligned}
\mathcal{F}_c[f'] &= -\frac{1}{\omega}\mathcal{F}_s[f''] \\
&= -\frac{1}{\omega}\left(\frac{2}{\pi}\omega f(0) - \omega^2\mathcal{F}_s[f] \right) \\
&= -\frac{2}{\pi}f(0) + \omega\mathcal{F}_s[f].
\end{aligned}$$

4. We use the first and third identities:

$$\begin{aligned}
\mathcal{F}_c[f''] &= -\frac{2}{\pi} f'(0) + \omega \mathcal{F}_s[f'] \\
&= -\frac{2}{\pi} f'(0) + \omega \left(-\omega \mathcal{F}_c[f]\right) \\
&= \frac{-2}{\pi} f'(0) - \omega^2 \mathcal{F}_c[f].
\end{aligned}$$

20.5 Problem set 5

Exercise. 1. Problem 1, Lesson 11, Farlow.

What is the Fourier series expansion of the square sine wave

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 \le x < 1 \end{cases}$$
$$f(x+2) = f(x) \qquad \text{(periodic condition)}$$

Graph the first 2,3,4 terms of the series to see how it is converging to $f(x)$. Also graph the frequency spectrum of $f(x)$.

Solution. 1. $f(x)$ can be represented by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right],$$

where

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)\,dx \\ &= \int_0^1 \cos(n\pi x)\,dx + \int_{-1}^{0} (-1)\cos(n\pi x)\,dx \\ &= 0. \end{aligned}$$

and

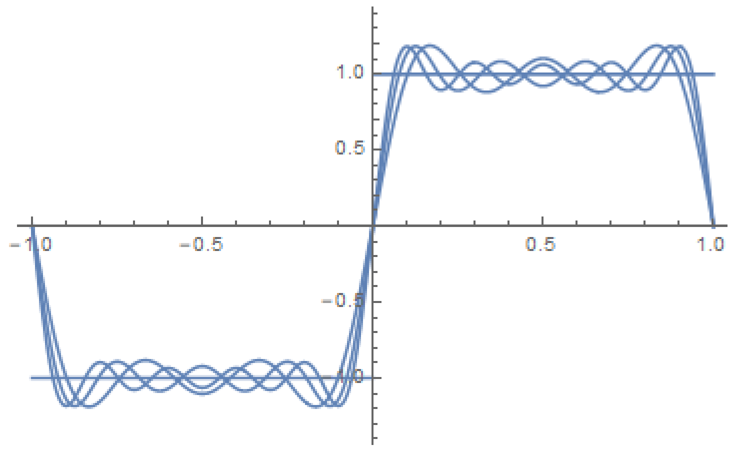
$$\begin{aligned} b_n &= \int_0^1 \sin(n\pi x)\,dx + \int_{-1}^{0} (-1)\sin(n\pi x)\,dx \\ &= \frac{2 - 2\cos(\pi n)}{\pi n} \\ &= \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & n \text{ odd} \end{cases}. \end{aligned}$$

So, the Fourier series expansion of $f(x)$ is

$$f(x) = \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\left((2j+1)\pi x\right).$$

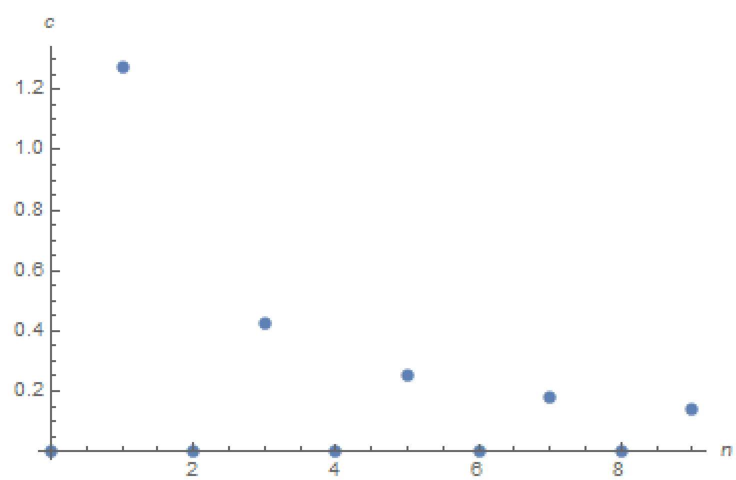
Graph of the first 2,3,4 terms of the series:

```
ffs[x_, nmax_] := Sum[4/Pi/(2 n + 1)*Sin[(2 n + 1) Pi*x], {n, 0, nmax}]

pn[n_] := Plot[ffs[x, n], {x, -1, 1}]

Show[p, pn[2], pn[3], pn[4], PlotRange -> {-1.3, 1.3}]
```



Frequency spectrum of $f(x)$:

```
data := Table[{n, (2 - 2 Cos[Pi*n])/(n*Pi)}, {n, 0.000000000001, 10}]

Show[ListPlot[data], AxesLabel -> {n, c}]
```



Exercise. 2. Problem 2, Lesson 11, Farlow.

Show that if we differentiate the Fourier series expansion of the sawtooth wave term by term, we arrive at an infinite series that clearly does not represent the derivative of the sawtooth curve.

Solution. 2. The Fourier series expansion of the sawtooth wave is

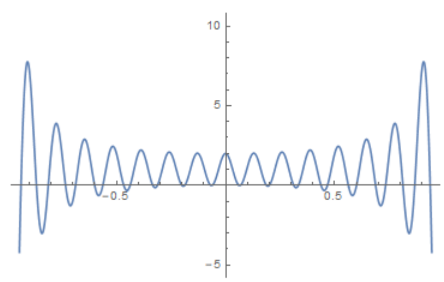
$$f(x) = \frac{2L}{\pi}\left[\sin(\pi x/L) - \frac{1}{2}\sin(2\pi x/L) + \frac{1}{3}\sin(3\pi x/L) - \dots\right].$$

Differentiating $f(x)$ with respect to x gives

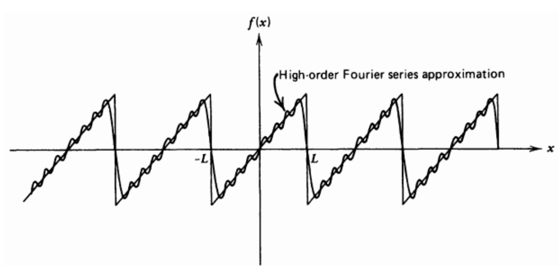
$$\begin{aligned}\frac{df(x)}{dx} &= \frac{2L}{\pi}\left[\frac{\pi}{L}\cos(\pi x/L) - \frac{1}{2}\frac{2\pi}{L}\cos(2\pi x/L) + \frac{1}{3}\frac{3\pi}{L}\cos(3\pi x/L) - \dots\right] \\ &= 2\left[\cos(\pi x/L) - \cos(2\pi x/L) + \cos(3\pi x/L) - \dots\right],\end{aligned}$$

which looks like (taking $L = 1$)

```
ffs2[x_, nmax_] := 2*Sum[Cos[n*Pi*x]*(-1)^(n + 1), {n, 1, nmax}]
pn2[n_] := Plot[ffs2[x, n], {x, -1, 1}]
Show[pn2[15], PlotRange -> {10, -5}]
```



whereas the derivative of the sawtooth curve given in the book:



is given by

$$\begin{aligned}f'(x) &= 1, \qquad -L < x < L \\ f'(x + 2L) &= f(x), \qquad \text{(periodic condition)}\end{aligned}$$

whose Fourier series expansion is just a constant.

Exercise. 3. Problem 3, Lesson 12, Farlow.

Solve the IVP:

$$\begin{aligned} PDE: u_t = \alpha^2 u_{xx} \qquad & -\infty < x < \infty \\ IC: u(x,0) = e^{-x^2} \qquad & -\infty < x < \infty \end{aligned}$$

by using the Fourier transform.

Solution. 3. We first transform the problem:

$$\begin{aligned} \mathcal{F}_x[u_t](\xi) &= \alpha^2 \mathcal{F}_x[u_{xx}](\xi) \\ \mathcal{F}_x[u(x,0)](\xi) &= \mathcal{F}_x\left[e^{-x^2}\right](\xi). \end{aligned}$$

Let $U(\xi,t) = \mathcal{F}_x[u(x,t)](\xi)$. Since we are working with the Fourier transform in x, we can take the $t-$derivative outside of the FT to get an ODE:

$$\mathcal{F}_x[u_t(x,t)](\xi) = \frac{d\mathcal{F}_x[u(x,t)](\xi)}{dt} = \alpha^2 \mathcal{F}_x[u_{xx}](\xi) = \boxed{\frac{dU(\xi,t)}{dt} = -\alpha^2\xi^2 U(\xi,t)}$$

FT the IC gives:

$$U(\xi,0) = \mathcal{F}_x\left[e^{-x^2}\right](\xi).$$

Solving the ODE gives:

$$U(\xi,t) = U(\xi,0)e^{-\alpha^2\xi^2 t}.$$

Now, we take the inverse:

$$\begin{aligned} u(x,t) &= \mathcal{F}_x^{-1}[U(\xi,t)] \\ &= \mathcal{F}_x^{-1}\left[U(\xi,0)e^{-\alpha^2\xi^2 t}\right] \\ &= \mathcal{F}_x^{-1}[U(\xi,0)] * \mathcal{F}_x^{-1}[e^{\alpha^2\xi^2 t}] \\ &= u(x,0) * \mathcal{F}_x^{-1}[e^{-\alpha^2\xi^2 t}], \qquad \text{convolution theorem} \\ &= e^{-x^2} * \mathcal{F}_x^{-1}[e^{-\alpha^2\xi^2 t}] \\ &= e^{-x^2} * \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha^2\xi^2 t} e^{ix\xi}\, d\xi\right]. \end{aligned}$$

Change of variables: let $s = -\alpha\sqrt{t}\xi$, so that $ds = -\alpha\sqrt{t}$, we have

$$\begin{aligned}
u(x,t) &= e^{-x^2} * \left[\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha\sqrt{t}}\int_{-\infty}^{\infty} e^{-s^2} e^{-ix(s/\alpha\sqrt{t})}\,ds\right] \\
&= e^{-x^2} * \frac{1}{\alpha\sqrt{t}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-s^2} e^{-is(x/\alpha\sqrt{t})}\,ds\right] \\
&= e^{-x^2} * \frac{1}{\alpha\sqrt{t}}\mathcal{F}[e^{-s^2}\left(\frac{x}{\alpha\sqrt{t}}\right)] \\
&= e^{-x^2} * \frac{1}{\alpha\sqrt{t}}\frac{1}{\sqrt{2}}e^{-(x/\alpha\sqrt{t})^2/2^2} \\
&= e^{-x^2} * \frac{1}{\alpha\sqrt{2t}}e^{-x^2/4\alpha^2 t} \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha\sqrt{2t}}\int_{-\infty}^{\infty} e^{-y^2} e^{-(x-y)^2/4\alpha^2 t}\,dy \\
&= \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2} e^{-(x-y)^2/4\alpha^2 t}\,dy.
\end{aligned}$$

So the solution to the IVP is

$$\boxed{u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2} e^{-(x-y)^2/4\alpha^2 t}\,dy}$$

Exercise. 4. Problem 5, Lesson 12, Farlow.

Verify that the convolution of two functions f and g can be written as either

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\, d\xi$$

or

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\, d\xi.$$

Solution. 4. We can show the two expressions are equivalent by change of variables. Let

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\, d\xi$$

and $y = x - \xi$. Then $\xi = x - y$ and $d\xi = -dy$. So,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\, d\xi &= \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} -f(x-y)g(y)\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\, d\xi,
\end{aligned}$$

where we just replace the dummy variable y with ξ. Therefore, the two expressions are equivalent. This also shows that $(f * g)(x) = (g * f)(x)$.

Exercise. 5. Problem 2, Lesson 12, Farlow.

Verify that the Fourier and inverse Fourier transforms are linear transformations.

Solution. Let "nice" functions $f(x)$ and $g(x)$ and $\alpha, \beta \in \mathbb{C}$ be given.

1. The Fourier transform is linear.

$$\begin{aligned}
\alpha\mathcal{F}[f(x)](\xi) + \beta\mathcal{F}[g(x)](\xi) &= \alpha\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f(x))e^{-ix\xi}\,dx + \beta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(g(x))e^{-ix\xi}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\alpha f(x))e^{-ix\xi}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\beta g(x))e^{-ix\xi}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}(\alpha f(x) + \beta g(x))e^{-ix\xi}\,dx\right) \\
&= \mathcal{F}[\alpha f + \beta g](\xi).
\end{aligned}$$

2. The Inverse Fourier transform is also linear.

$$\begin{aligned}
\alpha\mathcal{F}^{-1}[f(\xi)](x) + \beta\mathcal{F}^{-1}[g(\xi)](x) &= \alpha\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\xi)e^{ix\xi}\,d\xi + \beta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(\xi)e^{ix\xi}\,d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\alpha f(\xi)e^{ix\xi}\,d\xi + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\beta g(\xi)e^{ix\xi}\,d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\alpha f(\xi) + \beta g(\xi))e^{ix\xi}\,d\xi \\
&= \mathcal{F}^{-1}[\alpha f(\xi) + \beta g(\xi)](x).
\end{aligned}$$

Exercise. 6. In this exercise, you'll derive the formula given as Item 3 in Table 12.1 in our textbook. The formula essentially says that the Fourier transform of a Gaussian function is another Gaussian function. Precisely, this is the identity

$$\mathcal{F}[e^{-x^2}](\xi) = \frac{1}{\sqrt{2}} e^{-(\xi/2)^2}$$

which holds for all $\xi \in \mathbb{R}$. To this end, do the following:

1. Fix any $x \in \mathbb{R}$ and define, for $t \in [0, 1]$,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+i(t/2)\xi)^2}\, dx.$$

Compute $f'(t)$ for $0 < t < 1$. Hint: You may differentiate right through the integral sign. In our answer, it is useful to keep the entire term which comes down from the chain rule together, i.e., keep your answer as a single integral.

Solution. 6.1 We simply follow what the hint says:

$$\begin{aligned}
\frac{df(t)}{dt} &= \frac{d}{dt}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+i(t/2)\xi)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dt} e^{-(x+i(t/2)\xi)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dt} e^{-(x+i(t/2)\xi)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\left(\frac{d}{dt}(x+i(t/2)\xi)^2\right) e^{-(x+i(t/2)\xi)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -2(x+i(t/2)\xi)\frac{i\xi}{2} e^{-(x+i(t/2)\xi)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -i\xi(x+i(t/2)\xi) e^{-(x+i(t/2)\xi)^2}\, dx.
\end{aligned}$$

2. By making a change of variables, $x \to y = x + i(t/2)\xi$, use symmetry to conclude that $f'(t) = 0$ for all $0 < t < 1$. Hint: In making the change of variables, you can pretend that the imaginary unit i can be treated as a real constant. Truthfully, this is not exactly correct but it works in this case. The true change of variables makes use of Cauchy's integral formula from complex analysis, which you will learn if you take MA352 in the fall.

Solution. 6.2 By change of variables: $y = x + i(t/2)\xi$, we have $dy = dx$, so:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -i\xi(x+i(t/2)\xi) e^{-(x+i(t/2)\xi)^2}\, dx = -\frac{i\xi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2}\, dy.$$

Since y is an odd function, e^{-y^2} is an even function, and that the domain of integration is symmetric, $f'(t) = 0$.

3. Recalling the result from introductory calculus that f is identically constant if and only if f' is identically zero, the previous step guarantees that $f(0) = f(1)$. Use this equation and the fact that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

which we derived in class to deduce the formula on top.

Solution. 6.3 Let $t = 0$, then

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2}}.$$

Since $f'(t) = 0$ for any $0 < t < 1$, we have $f(1) = f(0)$, so,

$$f(0) = f(1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+i(1/2)\xi)^2}\, dx = \frac{1}{\sqrt{2}}.$$

Hence,

$$\begin{aligned}
\mathcal{F}[e^{-x^2}](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+2i(1/2)\xi+(i\xi)^2/2^2)} e^{(i\xi)^2/2^2}\, dx \\
&= e^{-\xi^2/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+i(1/2)\xi)^2}\, dx \\
&= e^{-(\xi/2)^2} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+i(1/2)\xi)^2}\, dx \right) \\
&= e^{-(\xi/2)^2} f(1) \\
&= \frac{1}{\sqrt{2}} e^{-(\xi/2)^2}.
\end{aligned}$$

This verifies the given identity.

20.6 Problem set 6

Exercise. 1. Problem 3, Lesson 31, Farlow. What is Laplace's equation $\nabla^2 u = 0$ in polar coordinates if u depends only on r? What are the solutions of this equation? These are the *circularly symmetric potentials* in two dimensions.

Solution. 1. In polar coordinates, with $u = u(r)$,

$$\begin{aligned}\nabla^2 u &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ &= u_{rr} + \frac{1}{r}u_r.\end{aligned}$$

So the Laplace's equation becomes

$$u_{rr} + \frac{1}{r}u_r = 0.$$

Let $u_r = x(r)$, then $u_{rr} = x_r$ and

$$x_r + \frac{1}{r}x = 0.$$

Multiplying both sides by the integration factor $\exp\left[\int \frac{1}{r}\,dr\right] = r$, we get

$$(r \cdot x(r))' = 0,$$

which says $r \cdot x(r)$ is constant, i.e., $x(r) \propto \frac{1}{r}$, and thus

$$u(r) = \int x(r)\,dr = \int \frac{1}{r}\,dr = C_1 + C_2 \ln(r).$$

So the solution to the Laplace's equation in the circularly symmetric case is

$$\boxed{u(r) = C_1 + C_2 \ln(r)}$$

Exercise. 2. Problem 4, Lesson 31, Farlow. What is Laplace's equation in spherical coordinates if the solution u depends only on r? Can you find the solutions of this equation? These are the *spherically symmetric potentials* in three dimensions.

Solution. 2. In spherical coordinates, with $u = u(r)$,

$$\begin{aligned}\nabla^2 u &= u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta} \\ &= u_{rr} + \frac{2}{r}u_r.\end{aligned}$$

So the Laplace's equation becomes

$$u_{rr} + \frac{2}{r}u_r = 0.$$

Let $u_r = x(r)$, then $u_{rr} = x_r$ and

$$x_r + \frac{2}{r}x = 0.$$

Multiplying both sides by the integration factor $\exp\left[\int \frac{2}{r}\,dr\right] = r^2$, we get

$$(r^2 \cdot x(r))' = 0,$$

which says $r^2 \cdot x(r)$ is constant, i.e., $x(r) \propto \frac{1}{r^2}$, and thus

$$u(r) = \int x(r)\,dr = \int \frac{1}{r^2}\,dr = C_1 + C_2\frac{1}{r}.$$

So the solution to the Laplace's equation in the circularly symmetric case is

$$\boxed{u(r) = C_1 + C_2\frac{1}{r}}$$

Physically, this form resembles the gravitational and Coulomb's potentials.

Exercise. 3. Problem 1, Lesson 32, Farlow. Based on intuition, can you find the solution to the Dirichlet problem

$$\begin{aligned} \text{PDE: } \nabla^2 u = 0, &\quad 0 < r < 1 \\ \text{BC: } u(1,\theta) = \sin\theta, &\quad 0 \leq \theta < 2\pi? \end{aligned}$$

Solution. 3. Thinking in terms of separation of variables, $u(r,\theta) = R(r)\Theta(\theta)$, we can guess that $\Theta(\theta)$ has to be periodic like $\sin(\theta)$. Next, we can think of $R(r)$ as a function that scales $\Theta(\theta)$ so that at $r = 1$, the maximum of $u(r,\theta)$ is 1 (at $\theta = \pi/2$) and -1 at $\theta = 3\pi/2$. So we can guess that $u(r,\theta) = r\sin\theta$. It is first clear that $u(1,\theta) = \sin\theta$. If $\nabla^2 u = 0$ then we are done:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 + \frac{1}{r}\sin\theta - \frac{1}{r^2}r\sin\theta = 0.$$

So the solution is indeed

$$\boxed{u(r,\theta) = r\sin\theta}$$

Exercise. 4. Problem 2, Lesson 32, Farlow. Does the following Neumann problem

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 < r < 1$$
$$\text{BC: } \frac{\partial u}{\partial r} = \sin^2\theta$$

have a solution inside the circle?

Solution. 4. We check the compatibility condition for the Neumann problem:

$$\int_{\partial\Omega} g\,ds = \int_0^{2\pi} \sin^2\theta\,d\theta = 2\pi - \int_0^{2\pi} \cos^2\theta\,d\theta = 2\pi - \int_0^{2\pi} \sin^2\theta\,d\theta = \pi \neq 0.$$

Since the compatibility condition fails to hold, the Neumann problem fails to have a solution. Of course, we could have told by noticing that $\sin^2\theta$ is non-negative for any $\theta \in [0, 2\pi)$.

Exercise. 5. Problem 5, Lesson 32, Farlow. Now that you know the physical interpretation of the Laplacian, what is the general nature of solutions to the Helmholtz BVP

$$\text{PDE: } \nabla^2 u = -\lambda^2 u, \quad 0 < r < 1$$
$$\text{BC: } u(1, \theta) = 0, \quad 0 \le \theta < 2\pi$$

Solution. 5. The Laplacian has the "avering property," which says that if $\nabla^2 u < 0$ then the value of u at a point is greater than the average of u among its neighboring points, and if $\nabla^2 u > 0$ then the value of u at a point is less than the average of u among its neighboring points. For the PDE:

$$\nabla^2 u = -\lambda^2 u,$$

assuming that $\lambda \in \mathbb{R}$, we know that if u at a point is positive, then $\nabla^2 u < 0$, which means the average value of u near that point is less than u, then the neighboring points will have the tendency to "pull u down." On the other hand, if u is negative, then for the same reason the surround points will have the tendency to "pull u up." But we also have that $u(1, \theta) = 0$ on the boundary, this means this "oscillating" behavior is dampened and set to zero at the boundary.

So it makes sense that the solutions to this Helmholtz BVP are linear combinations of the Bessel functions each of which can look like

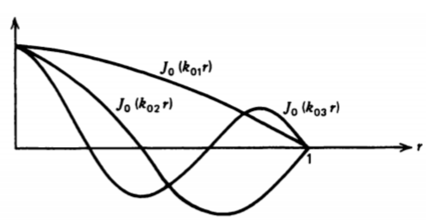


FIGURE 30.5 Graphs of $J(k_{0m}r)$ (basic building blocks of the vibrating membrane).

Exercise. 6. Problem 2, Lesson 33, Farlow. What is the solution to the interior Dirichlet problem

$$\text{PDE} : u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad 0 < r < 1$$

with the following BCs:

1. $u(1, \theta) = 1 + \sin\theta + \frac{1}{2}\cos\theta$
2. $u(1, \theta) = 2$
3. $u(1, \theta) = \sin\theta$
4. $u(1, \theta) = \sin 3\theta$

What do solutions look like? Do they satisfy Laplace's equation?

Solution. 6. We know that the general solution to the interior Dirichlet problem looks like

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n \left[A_n \cos(n\pi) + B \sin(n\pi)\right]$$

where A_n, B_n are obtained from the sine and cosine series expansion of $g(\theta)$.

1. Here, $A_0 = 1, B_0 = 0, B_1 = 1, A_1 = 1/2$ are the only non-zero coefficients. So, the solution is

   $$u(r, \theta) = 1 + r\sin\theta + \frac{1}{2}r\cos\theta.$$

   We should check that this solution satisfies the Laplace's equation:

   $$\begin{aligned} \nabla^2 u &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ &= 0 + \frac{1}{r}\left(\sin\theta + \frac{1}{2}\cos\theta\right) + \frac{1}{r^2}\left(-r\sin\theta - \frac{1}{2}r\cos\theta\right) \\ &= 0. \end{aligned}$$

2. Here, only $A_0 = 2$ is a non-zero coefficient, so the solution is just

   $$u(r, \theta) = 2.$$

   And of course this satisfies Laplace's equation:

   $$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

3. Here $B_1 = 1$ is the only non-zero coefficient. So the solution is

$$u(r,\theta) = r\sin\theta.$$

We check that this solution satisfies the Laplace's equation:

$$\begin{aligned}\nabla^2 u &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ &= 0 + \frac{1}{r}\sin\theta - \frac{1}{r^2}r\sin\theta \\ &= 0.\end{aligned}$$

In fact, this is Problem 1, Lesson 32, Farlow where we guessed the solution.

4. Here $B_3 = 1$ is the only non-zero coefficient. So the solution is

$$u(r,\theta) = r^3\sin 3\theta.$$

We should check that this satisfies the Laplace's equation:

$$\begin{aligned}\nabla^2 u &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ &= 6r\sin 3\theta + \frac{1}{r}(3r^2)\sin 3\theta + \frac{1}{r^2}r^3(-9)\sin 3\theta \\ &= (6+3-9)r\sin 3\theta \\ &= 0.\end{aligned}$$

Exercise. 7. Problem 5, Lesson 33, Farlow. Solve

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 < r < 1$$
$$\text{BC: } u(1,\theta) = \begin{cases} \sin\theta, & 0 \le \theta < \pi \\ 0, & \pi \le \theta < 2\pi \end{cases}$$

Roughly, what does the solution look like?

Solution. 7. We know that the general solution to the interior Dirichlet problem looks like

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n \left[A_n \cos n\theta + B_n \sin n\theta\right],$$

where

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\, d\phi = \frac{1}{2\pi}\int_0^{\pi} \sin\phi\, d\phi = \frac{1}{\pi}.$$

Next we evaluate coefficients A_n at $n \neq 0$:

$$\begin{aligned}
A_n &= \frac{1}{\pi}\int_0^{2\pi} \cos(n\phi) g(\phi)\, d\phi \\
&= \frac{1}{\pi}\int_0^{\pi} \cos(n\phi)\sin\phi\, d\phi \\
&= \frac{1}{2\pi}\int_0^{\pi} \sin(1+n)\phi + \sin(1-n)\phi\, d\phi \\
&= \frac{1}{2\pi}\left(\frac{1-\cos(n+1)\pi}{1+n} + \frac{1-\cos(1-n)\pi}{1-n}\right) \text{ if } n \neq 1, \qquad = 0 \text{ if } n = 1 \\
&= \frac{1}{2\pi}\left(\frac{1-\cos(n+1)\pi}{1+n} + \frac{1-\cos(n-1)\pi}{1-n}\right) \\
&= \frac{1}{2\pi}\left(\frac{1-\cos(n+1)\pi}{1+n} + \frac{1-\cos(n+1)\pi}{1-n}\right) \\
&= \frac{1}{2\pi}\left(\frac{2-2\cos(n+1)\pi}{1-n^2}\right) \\
&= \frac{1-\cos(n+1)\pi}{1-n^2} \\
&= \frac{1+\cos n\pi}{\pi(1-n^2)} \\
&= \begin{cases} 0, & n \text{ odd} \\ \frac{2}{\pi(1-n^2)}, & n \text{ even} \end{cases}
\end{aligned}$$

Finally, we evaluate coefficients B_n at $n \leq 0$:

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_0^{2\pi} \sin(n\phi) g(\phi)\, d\phi \\
&= \frac{1}{\pi}\int_0^{\pi} \sin(n\phi)\sin\phi\, d\phi \\
&= \frac{1}{2\pi}\int_0^{\pi} \cos(n-1)\phi - \cos(n+1)\phi\, d\phi \\
&= \frac{1}{2\pi}\left(\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{n+1}\right) \text{ if } n \neq \pm 1, \qquad = \pm 1/2 \text{ if } n = \pm 1 \\
&= \frac{1}{2\pi}\left(\frac{-\sin n\pi}{n-1} + \frac{\sin n\pi}{n+1}\right) \\
&= -\frac{\sin n\pi}{\pi(n^2-1)} \\
&= \begin{cases} 1/2, & n = 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

So, putting everything together, the general solution is

$$\boxed{u(r,\theta) = \frac{1}{2} r\sin\theta + \frac{2}{\pi}\left(\frac{1}{2} + \sum_{j=1}^{\infty} \frac{r^{2j}}{(1-(2j)^2)}\cos(2j\theta)\right)}$$

We can expand this to see a few terms...

$$u(r,\theta) = \frac{1}{2} r\sin\theta + \frac{2}{\pi}\left(\frac{1}{2} - \frac{r^2}{3}\cos 2\theta - \frac{r^4}{15}\cos 4\theta - \dots\right)$$

Exercise. 8. Problem 8, Lesson 33, Farlow. What does the Poisson kernel look like as a function of $a\alpha : 0 \leq \alpha < 2\pi$ for $r = 3R/4$, $\theta = \pi/2$? In other words, draw the graph of the Poisson kernel.

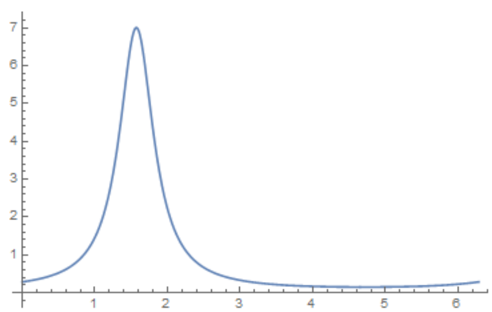
Solution. 8. The Poisson kernel is given by

$$P = \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \alpha) + r^2}.$$

At $\theta = \pi/2$, $r = 3R/4$, we have

$$\begin{aligned} P &= \frac{R^2 - (3R/4)^2}{R^2 - 2(3R/4)R\cos(\pi/2 - \alpha) + (3R/4)^2} \\ &= \frac{7/16}{1 - (3/2)\sin\alpha + 9/16} \\ &= \frac{7}{25 - 24\sin\alpha}. \end{aligned}$$

The graph:



Exercise. 9. Problem 8, Lesson 33, Farlow. Verify this: We can solve the BVP (nonhomogeneous BC)

$$\begin{aligned} \text{PDE: } \nabla^2 u = 0 \quad & \text{inside } D \\ \text{BC: } u = f \quad & \text{on the boundary of } D \end{aligned}$$

by

1. Finding any function V that satisfies the BC: $V = f$ on the boundary of D.
2. Solving the new BVP:

$$\begin{aligned} \text{PDE: } \nabla^2 W = \nabla^2 V \quad & \text{inside } D \\ \text{BC: } W = 0 \quad & \text{on the boundary of } D \end{aligned}$$

3. Observing that $u = V - W$ is the solution to our problem.

In other words, we can transform the nonhomogeneity from the BC to the PDE.

Solution. 9. For $V = f$ on the boundary of D, we have $V - u = 0$ on the boundary of D. Call this W, then $W = 0$ on the boundary of D. Take the Laplacian of $V - u$ inside D, and use the linearity of this operator to get

$$\nabla^2(V - u) = \nabla^2 V - \nabla^2 u = \nabla^2 V = 0 = \nabla^2 W.$$

Thus, we indeed end up with a new BVP with homogeneous BC:

$$\begin{aligned} \text{PDE: } \nabla^2 W = \nabla^2 V \quad & \text{inside } D \\ \text{BC: } W = 0 \quad & \text{on the boundary of } D \end{aligned}$$

20.7 Problem set 7

Exercise. 1. Problem 1, Lesson 34 Solve the Dirichlet problem

$$\text{PDE: } \nabla^2 u = 0 \qquad 1 < r < 2$$

$$\text{BCs: } \begin{cases} u(1,\theta) = \cos\theta \\ u(2,\theta) = \sin\theta \end{cases}$$

Solution. 1. We know that solutions have the form:

$$\begin{aligned} u(r,\theta) = &\sum_{n=0}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) b_0 \ln_n + \\ &+ \sum_{n=1}^{\infty} \alpha_n r^{-n} \cos(n\theta) + \beta_n r^{-n} \sin(n\theta). \end{aligned}$$

Solve for the $n = 0$ cases:

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, d\theta = 0 = a_0 + b_0 \ln(1) = a_0 \\ \frac{1}{2\pi}\int_0^{2\pi} \sin\theta\, d\theta = 0 = a_0 + b_0 \ln(2) = b_0 \ln(2), \end{aligned}$$

which says,

$$\begin{aligned} a_0 &= 0 \\ b_0 &= 0. \end{aligned}$$

Then we solve for a_n, α_n with $n \neq 0$:

$$\begin{aligned} \frac{1}{\pi}\int_0^{2\pi} \cos(\theta)\cos(n\theta)\, d\theta = a_n R_1^n + \alpha_n R_1^{-n} = a_n + \alpha_n \\ \frac{1}{\pi}\int_0^{2\pi} \sin(\theta)\cos(n\theta)\, d\theta = a_n R_2^n + \alpha_n R_2^{-n} = a_n 2^n + \alpha_n 2^{-n}. \end{aligned}$$

If $n = 1$ then

$$\begin{aligned} a_1 + \alpha_1 = 1 \\ 2a_1 + \frac{1}{2}\alpha_1 = 0 \end{aligned}$$

then

$$\begin{aligned} a_1 &= -\frac{1}{3} \\ \alpha_1 &= \frac{4}{3}. \end{aligned}$$

If $n \neq 1$, then

$$a_n + \alpha_n = 0$$
$$2^n a_n + \frac{1}{2^n}\alpha_n = 0$$

then

$$a_n = 0$$
$$\alpha_n = 0.$$

Then we solve for b_n, β_n with $n \neq 0$:

$$\frac{1}{\pi}\int_0^{2\pi} \cos(\theta)\sin(n\theta)\,d\theta = b_n + \beta_n.$$
$$\frac{1}{\pi}\int_0^{2\pi} \sin(\theta)\sin(n\theta)\,d\theta = b_n 2^n + \beta_n 2^{-n}.$$

If $n = 1$ then

$$b_1 + \beta_1 = 0$$
$$2b_1 + \frac{1}{2}\beta_1 = 1$$

then

$$b_1 = \frac{2}{3}$$
$$\beta_1 = -\frac{2}{3}.$$

If $n \neq 1$, then

$$b_n + \beta_n = 0$$
$$2^n b_n + \frac{1}{2^n}\beta_n = 0$$

then

$$b_n = 0$$
$$\beta_n = 0.$$

So the solution is

$$\boxed{u(r,\theta) = \left(-\frac{r}{3} + \frac{4}{3r}\right)\cos\theta + \left(\frac{2r}{3} - \frac{2}{3r}\right)\sin\theta}$$

□

Exercise. 2. Problem 2, Lesson 34 What is the solution to the exterior Dirichlet problem

$$\text{PDE: } \nabla^2 u = 0 \qquad 1 < r < \infty$$

for the following BCs:

1. $u(1,\theta) = 1$.
2. $u(1,\theta) = 1 + \cos(3\theta)$.
3. $u(1,\theta) = \sin(\theta) + \cos(3\theta)$.
4. $u(1,\theta) = \begin{cases} 1 & 0 \le \theta < \pi \\ 0 & \pi \le \theta < 2\pi \end{cases}$.

Solution. 2. Since the problem is on the exterior, the general solution is (after rejecting non-physical solutions)

$$u(r,\theta) = \sum_{n=0}^{\infty} \alpha_n r^{-n}\cos(n\theta) + \beta_n r^{-n}\sin(n\theta).$$

Plugging in $r = 1$,

$$u(1,\theta) = \sum_{n=1}^{\infty} \alpha_n \cos(n\theta) + \beta_n \sin(n\theta),$$

and observe that the given BCs are already in Fourier Sine/Cosine series, we get

1. $u(1,\theta) = 1 \implies u(r,\theta) = \boxed{1}$.
2. $u(1,\theta) = 1 + \cos(3\theta) \implies u(r,\theta) = \boxed{1 + \frac{1}{r^3}\cos(3\theta)}$.
3. $u(1,\theta) = \sin(\theta) + \cos(3\theta) \implies u(r,\theta) = \boxed{\frac{1}{r}\sin(\theta) + \frac{1}{r^3}\cos(3\theta)}$.
4. $u(1,\theta) = \begin{cases} 1 & 0 \le \theta < \pi \\ 0 & \pi \le \theta < 2\pi \end{cases}$, then

$$\begin{aligned}
\alpha_0 &= \frac{1}{2} \\
\beta_0 &= 0 \\
\alpha_n &= \frac{1}{\pi}\int_0^{\pi} \cos(n\theta)\, d\theta = 0 \\
\beta_n &= \frac{1}{\pi}\int_0^{\pi} \sin(n\theta)\, d\theta = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}
\end{aligned}$$

So,

$$\boxed{u(r,\theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{j=0}^{\infty} \frac{1}{(2j+1)r^{(2j+1)}} \sin((2j+1)\theta)}$$

$\square$

Exercise. 3. Problem 3, Lesson 34 The exterior Neumann problem

$$\text{PDE: } \nabla^2 u = 0 \qquad 1 < r < \infty$$
$$\text{BCs: } \frac{\partial u}{\partial r}(1,\theta) = g(\theta) \qquad 0 \le \theta \le 2\pi$$

has a solution that is the same form as the Dirichlet problem

$$u(r,\theta) = \sum_{n=0}^{\infty} r^{-n}\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

but now the coefficients a_n, b_n must satisfy the new BC. Substitute this solution in the BC

$$\frac{\partial u}{\partial r}(1,\theta) = \sin\theta$$

in order to obtain the solution to

$$\nabla^2 u = 0 \qquad 1 < r < \infty$$
$$\frac{\partial u}{\partial r}(1,\theta) = \sin\theta$$

Does you solution check? Of course, once you have this solution, any constant plus this solution is also a solution.

Solution. 3. Assuming the solution is of the form:

$$u(r,\theta) = \sum_{n=0}^{\infty} r^{-n}\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right],$$

it follows that

$$\frac{\partial u}{\partial r}(r,\theta) = \sum_{n=0}^{\infty} -nr^{-(n+1)}\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

At $r = 1$,

$$\frac{\partial u}{\partial r}(1,\theta) = \sum_{n=0}^{\infty} -n\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] = \sin\theta.$$

It is clear that $b_1 = -1$, and other coefficients are zero. So,

$$\boxed{u(r,\theta) = -\frac{1}{r}\sin\theta}$$

It is easy to check that

$$\nabla^2(u + C) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 2\frac{1}{r^3}\sin\theta - \frac{1}{r^3}\sin\theta - \frac{1}{r^3}\sin\theta = 0.$$

And,

$$\frac{\partial(u+C)}{\partial r}(1,\theta) = \frac{1}{r^2}\sin\theta\bigg|_{r=1} = \sin\theta.$$

So my solution checks, and any constant plus this solution is also a solution. □

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Exercise. 4. Problem 1, Lesson 35 Substitute $R(r) = r^\alpha$ into Euler's equation

$$r^2R'' + 2rR' - n(n+1)R = 0$$

to find $\alpha = n, -(n+1)$.

Solution. 4. For $R(r) = r^\alpha$,

$$\alpha(\alpha-1)r^2r^{\alpha-2} + 2\alpha r r^{\alpha-1} - n(n+1)r^\alpha = 0.$$

So,

$$\alpha(\alpha-1) + 2\alpha - n(n+1) = \alpha^2 + \alpha - n(n+1) = 0.$$

The solutions to this quadratic equation are

$$\alpha = \frac{-1 \pm \sqrt{1+4n(n+1)}}{2} = \frac{-1 \pm \sqrt{(2n+1)^2}}{2} = \frac{-1 \pm (2n+1)}{2} = \boxed{n, -(n+1)}$$

□

Exercise. 5. Problem 2, Lesson 35 Make the change of variable $x = \cos\phi$ to change the old Legendre's equation in ϕ

$$[\sin(\phi)\Phi']' + n(n+1)\sin(\phi)\Phi = 0 \qquad 0 \le \phi \le \pi$$

to the new Legendre's equation in x

$$(1-x^2)\frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx} + n(n+1)\Phi = 0 \qquad -1 \le x \le 1.$$

Solution. 5. For $x = \cos\phi$,

$$\begin{aligned}
0 &= [\sin(\phi)\Phi']' + n(n+1)\sin(\phi)\Phi \\
&= \cos\phi\Phi' + \sin\phi\Phi'' + n(n+1)\sin\phi\Phi \\
&= \cos\phi\frac{dx}{d\phi}\frac{d\Phi}{dx} + \sin\phi\Phi'' + n(n+1)\sin\phi\Phi \\
&= -\cos\phi\sin\phi\frac{d\Phi}{dx} + \sin\phi\Phi'' + n(n+1)\sin\phi\Phi
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &= -\cos\phi\frac{d\Phi}{dx} + \frac{dx}{d\phi}\frac{d}{dx}\left(\frac{d\Phi}{dx}\frac{dx}{d\phi}\right) + n(n+1)\Phi \\
&= -\cos\phi\frac{d\Phi}{dx} + \sin\phi\frac{d}{dx}\left(\frac{d\Phi}{dx}\sin\phi\right) + n(n+1)\Phi \\
&= -\cos\phi\frac{d\Phi}{dx} + \sin^2\phi\frac{d^2\Phi}{dx^2} + \sin\phi\cos\phi\frac{1}{-\sin\phi}\frac{d\Phi}{dx} + n(n+1)\Phi \\
&= (1-\cos^2\phi)\frac{d^2\Phi}{dx^2} - 2\cos\phi\frac{d\Phi}{dx} + n(n+1)\Phi \\
&= (1-x^2)\frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx} + n(n+1)\Phi,
\end{aligned}$$

as desired. $\square$

20.8 Problem set 8